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CONTENTS

Checker Boards and Polyominoes . . . . . S. W. GOLOMB 675
Zero-Free Regions of Polynomials . . . . . V. F. COWLING AND W. J. THRON 682
Some Remarks on Stirling's Formula . . . . M. I. AISSEN 687
Mathematical Notes . O. BOTTEMA, E. P. MILES, H. L. ALDER, VICTOR THÉBAULT, T. S. NANJUNDIAH, M. H. PROTTER 692
Classroom Notes. M. S. KLAMKIN, ROGER OSBORN, A. J. COLEMAN 705
Elementary Problems and Solutions 711
Advanced Problems and Solutions 718
Recent Publications 723
News and Notices 726
The Mathematical Association of America 733
Itineraries of Visiting Lecturers. 733
The Fifteenth Annual William Lowell Putnam Mathematical Competition 734
Calendar of Future Meetings 736
Index to Volume 61 737

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CHECKER BOARDS AND POLYOMINOES

S. W. GOLOMB, Harvard University

Our starting point is the well-known problem: Given a checker board with a pair of opposite corners deleted (Fig. 1), and given a box of dominoes, where each domino covers exactly two squares of the checker board, is it possible to cover this checker board exactly with dominoes? The answer is "no"; for suppose that the checker board is colored in the usual manner (Fig. 1). Then each domino covers one light square and one dark square. Thus n dominoes would cover n light squares and n dark squares, that is, an equal number of each. But the checker board of Fig. 1 has more dark squares than light squares, and so it can not be covered with dominoes.

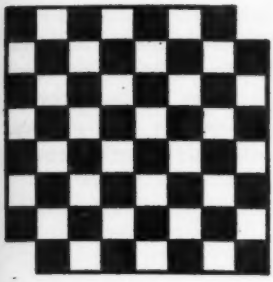

FIG. 1

We shall retain the 8×8 checker board as our "canonical domain," but we shall generalize the "domino" to the "polyomino," and our theorems will involve all the simpler polyominoes, shown in Figure 2. More precisely, we *define* an n-omino as a simply-connected set of n squares of the checker board which are "rook-wise connected"; that is, a rook placed at any square of the n-omino must be able to get to any other square, in a finite number of moves.

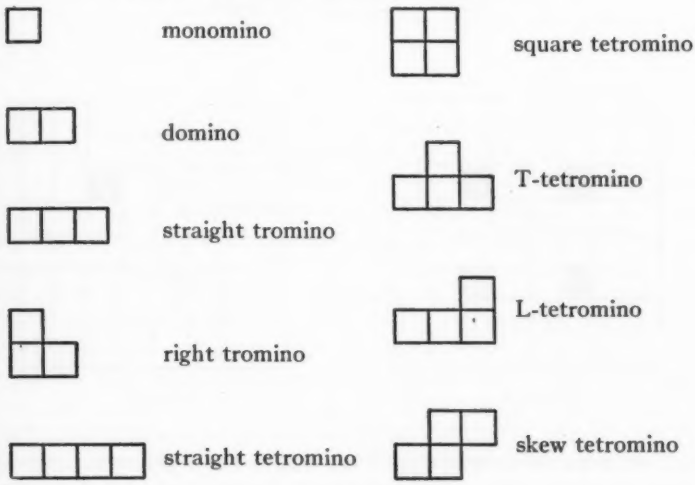

FIG. 2

First we consider trominoes. Clearly it is impossible to cover the 8×8

checker board with trominoes, because 64 is not divisible by 3. Instead, we will try to cover the checker board with 21 trominoes and one monomino.

It is impossible to cover the checker board with 21 *straight* trominoes, and a monomino in the upper left-hand corner of the board. To prove this, we color the checker board in three colors, *a*, *b*, and *c*, as shown in Figure 3. There are 21 *a*-colored squares, 22 *b*-colored squares, and 21 *c*-colored squares. Every straight tromino covers one *a*-colored square, one *b*-colored square, and one *c*-colored square, so that the straight trominoes will always cover an equal number of *a*-squares, *b*-squares, and *c*-squares. But the upper left-hand corner is an *a*-square, so that covering it with a monomino leaves only 20 *a*-squares, but 22 *b*-squares and 21 *c*-squares. These numbers are unequal; hence the covering with straight trominoes is impossible.

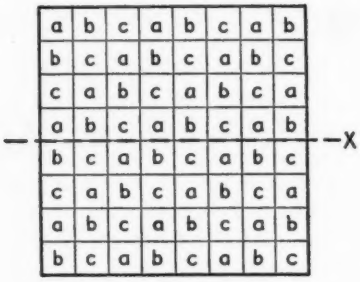

FIG. 3

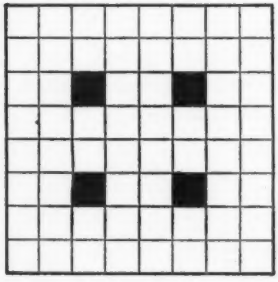

FIG. 4

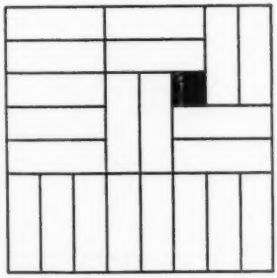

FIG. 5

The same argument shows that if the monomino is placed on any *a*-square, or on any *c*-square, the rest of the board cannot be covered with straight trominoes. Finally, suppose the monomino is placed on a *b*-square. For example, the lower left-hand corner. But by symmetry in the *x*-axis, this is really the same problem as when the monomino was in the *upper* left-hand corner, and

that covering was impossible. In fact, only *b*-squares which are symmetric to other *b*-squares are possible locations for the monomino. The only such squares are the four dark squares of Figure 4. And from Figure 5, we see that if the monomino is placed on one of these squares, the rest of the board really *can* be covered with straight trominoes.

Thus we have proved: *A necessary and sufficient condition for the checker board to be coverable with* 21 *straight trominoes and one monomino is that the monomino be placed on one of the dark squares of Figure 4.*

So far we have worked mainly with "non-existence" proofs. Our next result is in the opposite direction:

No matter where on the checker board a monomino is placed, the remaining squares can always be covered with 21 *right trominoes.*

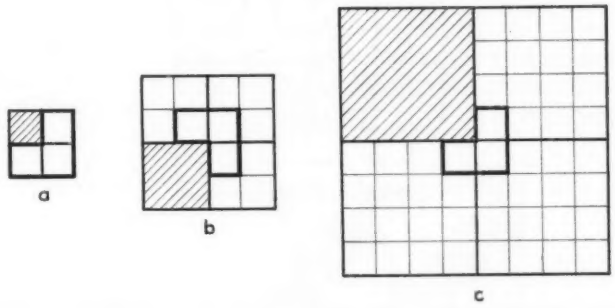

FIG. 6a–c

The proof proceeds by induction in $2^n \times 2^n$ checker boards. For a 2×2 checker board (Fig. 6-a), it is clear that wherever a monomino is placed, the rest can be covered by a right tromino. Given a 4×4 checker board, we divide it into quadrants (Fig. 6-b). Let a monomino be in one of these quadrants, the third, say. Each of the quadrants is a 2×2 checker board; so we can put a monomino in each quadrant, and cover the rest of the board with right trominoes. In the third quadrant the monomino is already assigned. In the other three quadrants, we place our monominoes in the center (Fig. 6-b); and then we can replace these three monominoes by a single tromino. Thus we have covered the 4×4 checker board with one monomino in a preassigned position, and 5 right trominoes. The 8×8 case, and indeed the general case, is treated in the same way (Fig. 6-c). We divide into quadrants; the preassigned monomino lies in one of these quadrants, and we can finish covering that quadrant with right trominoes by the previous case. In each of the other three quadrants, we introduce monominoes in the central squares. Again, by the previous case, the rest of these quadrants can be covered by right trominoes. And the monominoes in the three central squares can be replaced by a single right tromino.

This proof is similar to the Bolzano-Weierstrass Theorem in two dimensions. We keep subdividing into quadrants until we locate a monomino, our discrete analogue of a cluster point.

Now we come to the tetrominoes. It is easy to cover the checker board entirely with straight tetrominoes, or with square tetrominoes, or with T-tetrominoes, or with L-tetrominoes. This should be clear from Figure 7. On the other hand, it is impossible to cover the board entirely with skew tetrominoes. In particular, it is impossible to place skew tetrominoes on the board in such a way that even a single edge is completely covered. Here I leave the details as an exercise.

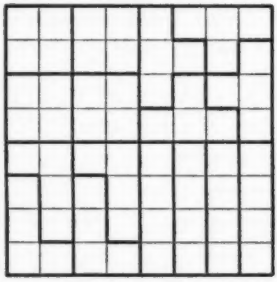

FIG. 7

Interesting tetromino problems are obtained in the following way: Suppose we try to cover the checker board with 15 T-tetrominoes and one square tetromino. But this is impossible. For consider the ordinary checker board coloring (*cf.* Fig. 1). Here a square tetromino covers two dark squares and two light squares—an even number of each. But a T-tetromino will cover three light and one dark square, or else three dark and one light square, in any case an odd number of each. There are 15 T-tetrominoes, an odd number, each covering an odd number of light and an odd number of dark squares. By a result of elementary arithmetic, an odd number of odd numbers is odd; by another result, an even number plus an odd number is odd. Thus 15 T-tetrominoes and one square tetromino cover an odd number of dark and an odd number of light squares. But with the ordinary checker board coloring, there are an even number of squares of each type.

A similar result holds for L-tetrominoes: ***The checker board cannot be covered with 15 L-tetrominoes and one square tetromino.*** To prove this, we introduce the coloring of Figure 8. Here, again, a square tetromino covers two light and two dark squares, while each L-tetromino covers three of one and one of the other, an odd number of each. Thus again 15 L-tetrominoes and a square tetromino will cover an odd number of dark and an odd number of light squares; but

clearly there are an even number of light and of dark squares in Figure 8.

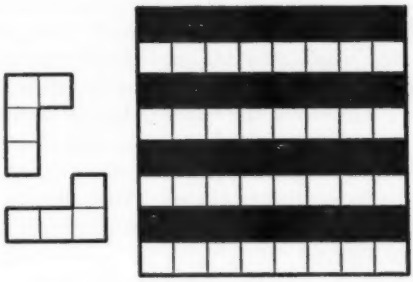

FIG. 8

We prove the corresponding result for both straight tetrominoes and skew tetrominoes in a single theorem: *The checker board cannot be covered with a square tetromino and* 15 *other tetrominoes of which some* (*or all*) *are straight and the rest are skew.*

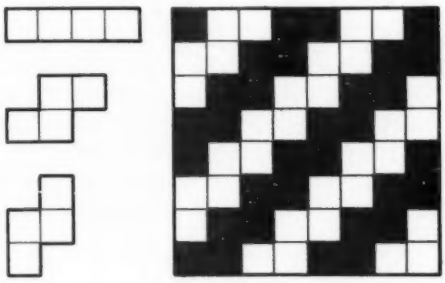

FIG. 9

This time we introduce the coloring of Figure 9. Here it is the *square* tetromino which covers an odd number of dark and an odd number of light squares, while straight tetrominoes always cover two of each, and skew tetrominoes cover either two of each, or four of one and none of the other. In any case, straight as well as skew tetrominoes cover an even number of dark and an even number of light squares. Adding on the *odd* number of squares of each color covered by the square tetromino, we find that the 16 tetrominoes in question always cover an odd number of dark and an odd number of light squares. But in each row of Figure 9 there are four light and four dark squares; hence the entire board consists of an *even* number of each, and cannot be covered with the given tetrominoes.

There are several directions in which we may now proceed.

1. *Problems on the torus.* If we bend the checker board around until the right and left sides meet, we get a cylinder. If we take this cylinder and bend it around until the top and bottom meet, we get a doughnut, or *torus*. Any covering with polyominoes which was possible before, on the ordinary board, is still possible on the torus. The impossibility theorems, however, must be re-examined.

We know that we can cover the checker board with a monomino and 21 straight trominoes (Fig. 5), provided the monomino is placed on a suitable square. On the torus, every square is just like every other, so that the covering is possible *wherever* the monomino is placed. (Note that on the torus, a straight tromino will *not* always cover one a-square, one b-square, and one c-square, in the coloring of Figure 3.)

The tetromino theorems are the same on the torus, because the various tetrominoes still cover the same number of light and dark squares as on the flat checker board, at least for the ordinary checker board coloring and the colorings of Figures 8 and 9. Moreover, we can use this fact to conclude something about the original checker board: Since the four corner squares of the checker board form a square tetromino on the torus, it is impossible to remove the four corner squares from a checker board and cover the rest with 15 T-tetrominoes, or with 15 L-tetrominoes, or with a combination of 15 straight and skew tetrominoes.

2. *Algebraic significance of the colorings.* Considering the squares of the checker board as lattice points in the plane, the light and dark squares of the ordinary coloring are the two residue classes of the Gaussian integers $a+bi$ modulo $1+i$. The coloring of Figure 3 shows the three residue classes of $a+b\omega$ modulo $1+\omega$, where ω is a primitive cube root of unity.

3. *Pseudo-polyominoes.* If we admit "queen-wise" as well as "rook-wise" connected polyominoes, we get the *pseudo-polyominoes*, the simplest of which appear in Figure 10.

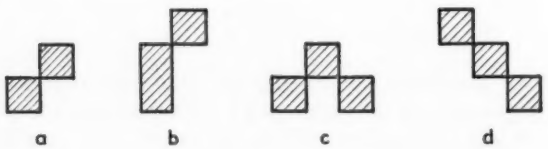

FIG. 10

I recommend the following problems:

(i) Show that the checker board cannot be covered with 21 pseudo-trominoes of types 10-c and 10-d, and one monomino.

(ii) Show that the checker board *can* be covered with 21 pseudo-trominoes of type 10-b and one monomino. Where must the monomino be placed?

(iii) Find all the pseudo-tetrominoes.

We can generalize even further, to the *quasi-polyomino*, which need not be connected at all. Figure 11 shows first a certain quasi-tromino; then it shows how two of these may be combined to form a certain hexomino; and using 10 of these hexominoes, with one of the original quasi-trominoes, and one monomino, how to cover the checker board. Thus it is possible to cover the checker board with 21 of these quasi-trominoes, and one monomino.

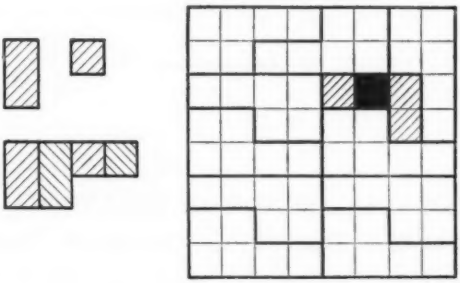

FIG. 11

4. *Pentominoes.* There are twelve distinct pentominoes. They all appear in Figure 12, which moreover solves the problem: Is it possible to place all 12 pentominoes on the checker board at the same time? The solution pictured here is the "best," in the sense that the four squares left over not merely form a tetromino, but a square tetromino, in the center of the board.

Other interesting pentomino problems arise when one tries to cover the checker board with 12 pentominoes of a single type, and one square tetromino.

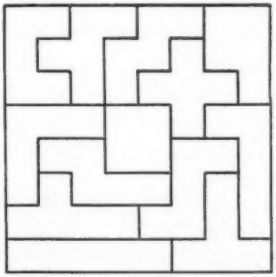

FIG. 12

5. *The board.* The shape of the checker board can be altered at will to obtain new problems. For example, is it possible to fit all twelve distinct pentominoes

on a 3×20 board? Also, polyomino problems in 3 or even more dimensions can be considered. However, I stated earlier that I would confine myself at present to the 8×8 checker board.

Another possible modification, of a rather fundamental sort, would be the use of hexagonal rather than square tiles for the board and for the objects covering it.

ZERO-FREE REGIONS OF POLYNOMIALS*

V. F. COWLING, University of Kentucky and W. J. THRON, Washington University

1. Introduction. In this paper we are concerned with the derivation of certain results about the location of the zeros of a polynomial. We use a method which was developed by Leighton and Thron [4] to study the convergence of continued fractions, was then adapted by Cowling [1] to obtain results on the distribution of values of the partial sums of a Taylor series, and has now been further modified by us to make it as suitable as possible for our present purpose. This method is developed in Section 2. The next two sections bring a number of applications of this method. The last section is devoted to a discussion of the relation of our results to other results of a similar nature.

2. Notation and basic considerations. Let V be a region in the complex plane; then $1/V$ and $V+a$ are, respectively, the regions consisting of all z such that $1/z\in V$ and all w such that $w-a\in V$. Further, throughout this paper indices k, m, p and q will be used to indicate the following ranges:

$$k = 1, \cdots, n; \quad m = 1, \cdots, n-1; \quad p = 0, \cdots, n-1; \quad q=0, \cdots, n.$$

Now let

$$P(z) = a_0 + a_1 z^{\lambda_1} + \cdots + a_n z^{\lambda_n}, \tag{2.1}$$

where all a_q are assumed to be different from zero, and where the λ_k are positive integers satisfying the relation $\lambda_1<\lambda_2< \cdots <\lambda_n$. It is convenient also to introduce $\lambda_0=0$. The integers α_k are to be defined as follows:

$$\alpha_k = \lambda_k - \lambda_{k-1}.$$

* This research was supported in part by the United States Air Force under contract No. AF 18(600)-419 monitored by the office of Scientific Research, Air Research and Development Command.

It is clear that

$$P(z) = a_0\left(1+\frac{a_1}{a_0}z^{\alpha_1}\left(1+\frac{a_2}{a_1}z^{\alpha_2}\left(1+\cdots\left(1+\frac{a_n}{a_{n-1}}z^{\alpha_n}\right)\cdots\right)\right)\right), \tag{2.2}$$

and the result below follows.

LEMMA 2.1. *Let two sets of regions $V_0, \cdots, V_{n-1}$ and $E_1, \cdots, E_n$ be given among which the following relations hold:*

$$1 + e_m v_m \in V_{m-1} \text{ for all } e_m \in E_m \quad \text{and} \quad v_m \in V_m, \quad 1 + E_n \subset V_{n-1}.$$

If

$$\frac{a_k}{a_{k-1}} z^{\alpha_k} \in E_k$$

for all k, then $P(z) \in a_0 V_0$.

In some cases, as for example in Section 4, the sets V_p can easily be computed from given sets E_k. Frequently, however, it is advisable to begin with a given set of V_p and construct the E_k from them. This construction is described below.

LEMMA 2.2. *Let regions $V_0, \cdots, V_{n-1}$ be given such that the regions*

$$E_m = \bigcap_{v_m \in V_m} \frac{1}{v_m}(V_{m-1} - 1),$$

$$E_n = V_{n-1} - 1,$$

are all non-empty. If

$$\frac{a_k}{a_{k-1}} z^{\alpha_k} \in E_k,$$

then $P(z) \in a_0 V_0$.

To prove this lemma it suffices to show that the E_k, constructed in the way described here, together with the V_p from which they are derived, satisfy the conditions of Lemma 2.1. Clearly If $E_n = V_{n-1} - 1$ then $1 + E_n \subset V_{n-1}$. For every $e_m \in E_m$ we have

$$e_m = (v'_{m-1} - 1)/v_m, \tag{2.3}$$

where v_m is an arbitrary element of V_m and $v'_{m-1} - 1$ is some element of V_{m-1}. Solving (2.3) for v'_{m-1} we obtain $v'_{m-1} = 1 + e_m v_m$ and hence $1 + e_m v_m \in V_{m-1}$ for all $e_m \in E_m$ and $v_m \in V_m$. We might observe at this point that indiscriminate choice of the V_p might lead to empty E_k and that to obtain regions E_k as large as possible, which is clearly our aim, requires a careful search for suitable regions V_p.

3. Circular value regions. As a first application of the methods outlined in

the preceding section we consider regions V_p defined by

$$|z| > r_p,$$

where the r_p are arbitrary positive numbers except for r_0 which we set equal to zero. E_m then are the regions defined by

$$|e| > (1 + r_{m-1})/r_m$$

while E_n is defined by $|e-1| > r_{n-1}$. However, for the sake of uniformity, we replace the last region by the slightly less inclusive one defined by $|e| > 1+r_{n-1}$. If we now introduce $r_n = 1$, we have for every k, $e \in E_k$ if $|e| > (1+r_{k-1})/r_k$. It follows that the polynomial (2.1) does not assume the value zero for those z for which

$$\left| \frac{a_k}{a_{k-1}} z^{\alpha_k} \right| > \frac{1 + r_{k-1}}{r_k},$$

that is, for

$$|z| > \max \left[\left(\frac{1 + r_{k-1}}{r_k} \left| \frac{a_{k-1}}{a_k} \right|^{1/\alpha_k} \right) \right].$$

THEOREM 3.1. *The polynomial* (2.1) *assumes all its zeros for*

$$|z| > \max \left[\left(\frac{1 + r_{k-1}}{r_k} \left| \frac{a_{k-1}}{a_k} \right| \right)^{1/\alpha_k} \right].$$

where $r_0 = 0$, $r_n = 1$ *and the remaining* r_m *are arbitrary positive numbers.*

A well known result (see Dieudonné [3, p. 18]) is obtained as a corollary of this theorem if we set $r_k = 1$.

COROLLARY 3.1. *The polynomial* (2.1) *assumes all its zeros for*

$$|z| \leqq \max \left[\left| \frac{a_0}{a_1} \right|^{1/\alpha_1}, \left(2 \left| \frac{a_1}{a_2} \right| \right)^{1/\alpha_2}, \cdots, \left(2 \left| \frac{a_{n-1}}{a_n} \right| \right)^{1/\alpha_n} \right].$$

The choice of the free constants in Theorem 3.1 is facilitated by the introduction of $g_k = (1+r_{k-1})/r_k$. Then we have

$$r_m = \left(\sum_{\nu=0}^{m-1} \prod_{\mu=1}^{\nu} g_\mu \right) \Big/ \prod_{\nu=1}^{m} g_\nu.$$

Here we use the convention that $\prod_{\mu=1}^{0} g_\mu = 1$. From these relations it follows that g_1 to g_{n-1} may be taken to be arbitrary positive numbers while g_n is determined, since

$$g_n = (1 + r_{n-1})/r_n = 1 + r_{n-1} = \left(\sum_{\nu=0}^{n-1} \prod_{\mu=1}^{\nu} g_\mu \right) \Big/ \prod_{\nu=1}^{n-1} g_\nu.$$

We can thus restate Theorem 3.1 in terms of g_m.

THEOREM 3.1′. *The polynomial* (2.1) *assumes all its zeros for*

$$|z| \leqq R_g = \max\left[\left(g_1\left|\frac{a_0}{a_1}\right|\right)^{1/\alpha_1}, \cdots, \left(g_{n-1}\left|\frac{a_{n-2}}{a_{n-1}}\right|\right)^{1/\alpha_{n-1}}, \left(\frac{\sum_{r=0}^{n-1}\prod_{\mu=1}^{r} g_\mu}{\prod_{r=1}^{n-1} g_r}\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}\right]$$

where $g_1, \cdots, g_{n-1}$ *are arbitrary positive numbers.*

In an application of this theorem the $|a_{k-1}/a_k|$ and the α_k would be given and it would be our task to select the g_m in such a way as to make R_g as small as possible. We would accordingly assign large values to those g_m whose corresponding $|a_{m-1}/a_m|$ are small. The obvious difficulty is that we have no free control over the coefficient of $|a_{n-1}/a_n|$. We illustrate the possible choices that can be made by a few examples. Taking $g_m=1$ we obtain

$$R_g = \max\left[\left|\frac{a_0}{a_1}\right|^{1/\alpha_1}, \cdots, \left|\frac{a_{n-2}}{a_{n-1}}\right|^{1/\alpha_{n-1}}, \left(n\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}\right].$$

If we let $g_m=1$, $m\neq r$, and $g_r=h$, we have

$$R_g = \max\left[\left|\frac{a_0}{a_1}\right|^{1/\alpha_1}, \cdots, \left(h\left|\frac{a_{r-1}}{a_r}\right|\right)^{1/\alpha_r}, \cdots, \left|\frac{a_{n-2}}{a_{n-1}}\right|^{1/\alpha_{n-1}}, \left(\frac{r+h(n-r)}{h}\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}\right].$$

If a_r differs markedly from the other coefficients a choice of $g_m=1$, $m\neq r, r+1$, $g_r=h$, $g_{r+1}=1/h$ might give a good value for R_g. It is

$$R_g = \max\left[\left|\frac{a_0}{a_1}\right|^{1/\alpha_1}, \cdots, \left(h\left|\frac{a_{r-1}}{a_r}\right|\right)^{1/\alpha_r}, \left(\frac{1}{h}\left|\frac{a_r}{a_{r+1}}\right|\right)^{1/\alpha_{r+1}}, \cdots \left|\frac{a_{n-2}}{a_{n-1}}\right|^{1/\alpha_{n-1}}, \left((n-1+h)\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}\right].$$

Finally the choice $g_m=|a_m/a_{m-1}|$ gives

$$R_g = \max\left[1, \left(\left(\sum_{r=0}^{n-1}|a_r|\right)\Big/|a_n|\right)^{1/\alpha_n}\right].$$

For $\lambda_q=q$ this is a known result (see [3, p. 17]).

The fact that the roots of the polynomial

$$Q(z) = a_n + a_{n-1}z^{\lambda_n-\lambda_{n-1}} + \cdots + a_1 z^{\lambda_n-\lambda_1} + a_0 z$$

are the reciprocals of the roots of the polynomial (2.1) leads immediately to the result below.

THEOREM 3.2. *The polynomial* (2.1) *assumes all its zeros for*

$$|z| \geqq \min\left[\left(f_1\left(\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}, \cdots, \left(\left|\frac{a_0}{a_1}\right|\Big/\left(\sum_{\nu=0}^{n-1}\prod_{\mu=1}^{\nu} f_\mu\right)\right)^{1/\alpha_1}\right]$$

where $f, \cdots, f_{n-1}$ *are arbitrary positive numbers.*

4. Angular openings as value regions. It follows immediately from formula (2.2) that requiring

$$|\arg((a_k/a_{k-1})z^{\alpha_k})| < \pi\alpha_k/\lambda_n$$

leads to V_p satisfying $0 \notin V_p$ and

$$|\arg z| < \pi(\lambda_n - \lambda_p)/\lambda_n.$$

The region V_0 thus does not contain that part of the real axis for which $R(z) \leqq 0$. It is then clear that under these conditions the polynomial $P(z)$ does not vanish. If we now restrict ourselves to $a_q > 0$ the condition imposed on $(a_k/a_{k-1})z^{\alpha_k}$ is satisfied provided $|\arg z| < \pi/\lambda_n$. We finally observe that $P(z)$ vanishes for $|\arg z| = \pi/\lambda_n$ if and only if $P(z) = a_0 + a_n z^{\lambda_n}$. The following theorem has now been proved.

THEOREM 4.1. *If in the polynomial* (2.1) *of degree* $\lambda_n \geqq 2$ *all coefficients are positive then* $P(z)$ *does not vanish for*

$$|\arg z| \leqq \pi/\lambda_n$$

unless it is of the form $a_0 + a_n z^{\lambda_n}$ *in which case it has two zeros satisfying* $|\arg z| = \pi/\lambda_n$.

This result is a special case of a more general criterion due to Obrechkoff [6] the proof of which is considerably more involved than the one presented here.

5. Comparison with other criteria. We have already pointed out that some of our results are equivalent to previously known ones. Another criterion essentially due to Fujiwara (see [3, p. 18] and [5, p. 105]) states that all roots of the polynomial (2.1) have an absolute value not greater than

$$R_3 = \max\left[(\delta_p |a_p/a_n|)^{1/(\lambda_n - \lambda_p)}\right],$$

where the δ_p are positive numbers which satisfy the condition $\sum_{p=0}^{n-1} 1/\delta_p = 1$ but are otherwise arbitrary. This criterion may be shown to be equivalent to our Theorem 3.1′ if the best values for the δ_p and g_m can be determined. Since this however is likely to be difficult and since the main importance of these criteria seems to lie in the relative ease with which one can get fair approximations for the upper bound of the roots of a given polynomial, our results appear to be of value. Other zero-free regions, which are not contained in or equivalent to known criteria, have been obtained by the use of our method. The deriva-

tions, however, become sufficiently more complicated that it seemed advisable not to include them here.

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SOME REMARKS ON STIRLING'S FORMULA

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Introduction. Stirling's Formula

$$n! \sim \sqrt{2\pi} n^{1/2} n^n e^{-n} \tag{1}$$

is often presented in various stages. Various methods are used to prove the weaker form of Stirling's Formula,

$$n! \sim C n^{1/2} n^n e^{-n} \tag{2}$$

for some unspecified non-zero constant C. Then a standard elementary argument is used to show that $C = \sqrt{2\pi}$. [*Cf.* Courant: *Differential and Integral Calculus*, pp. 225, 363.]

In several of these methods, relatively crude arguments establish some tenuous connection between $n!$ and $n^n e^{-n}$. Then slightly refined arguments display the role of the factor $n^{1/2}$. It is the purpose of this note to present two lemmas, either of which can be used to supply the factor $n^{1/2}$. The two lemmas have little in common except for this possible application. Neither seems to be in the literature, although one (Lemma I) is practically a folk result. The second is perhaps new and was first noticed by A. Novikoff and the author in a different connection.

1. First method. It is perhaps natural to compare $n!$ and n^n since each is the

product of n simple factors. For $n>1$

$$1\cdot 2\cdot 3\cdot\cdots n<n\cdot n\cdot\cdots\cdot n$$

and we obtain the elementary inequality

$$n!<n^n. \tag{3}$$

To study the relationship between $n!$ and n^n more closely, we consider the ratio

$$u_n=n^n/n!. \tag{4}$$

The sequence $\{u_n\}$ clearly increases since

$$\frac{u_{n+1}}{u_n}=\left(1+\frac{1}{n}\right)^n. \tag{5}$$

We assume that the students have been exposed to this important sequence $\{(1+1/n)^n\}$ and know that it has a limit. (If not, this would seem to be a natural way to introduce it, rather than: "Consider the sequence $\{1+1/n\}^n$"). Thus we are led to study the sequence

$$V_n=\frac{u^n}{e^n}=\frac{n^ne^{-n}}{n!}. \tag{6}$$

Since the power series for e^x has positive coefficients, its sum is greater than any single term and we have $e^n>n^n/n!$ and thus $V_n<1$. More precisely, we learn that

$$\frac{V_{n+1}}{V_n}=\frac{\left(1+\frac{1}{n}\right)^n}{e}. \tag{7}$$

Before we exploit the connection between $n!$ and n^ne^{-n} that we have established, we will state and prove the first of the Lemmas.

LEMMA I. *If*

$$y_{n+1}=y_n\left\{1+\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right\}$$

and $y_n\neq 0$ *for all* n, *then*

$$y_n\sim Cn^\alpha$$

for some non-zero constant C.

Proof. Let $W_n = n^{-\alpha} y_n$. Now

$$(n+1)^{-\alpha} = n^{-\alpha}\left(1+\frac{1}{n}\right)^{-\alpha} = n^{-\alpha}\left\{1-\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right\}.$$

Hence,

$$W_{n+1} = W_n\left(1+\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1-\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right) = 1+O\left(\frac{1}{n^2}\right).$$

Therefore the infinite product

$$\frac{W_2}{W_1}\cdot\frac{W_3}{W_2}\cdot\frac{W_4}{W_3}\cdots = \prod_1^\infty\left(1+O\left(\frac{1}{n^2}\right)\right)$$

converges to some non-zero constant C. But if $\lim_{n\to\infty} W_n = C$, then

$$y_n \sim Cn^\alpha.$$

Now

$$(1+x)^{1/x} = \exp.\left\{\frac{1}{x}\log(1+x)\right\}$$

$$= e\left\{1-\frac{1}{2}x+O(x^2)\right\}.$$

Hence:

$$\frac{\left(1+\frac{1}{n}\right)^n}{e} = 1-\frac{1}{2n}+O\left(\frac{1}{n^2}\right). \tag{8}$$

Combining (7), (8) and Lemma I, we obtain

$$V_n = \frac{n^n e^{-n}}{n!} \sim C' n^{-1/2}$$

for some non-zero constant C'. Hence, if $CC' = 1$, we get

$$n! \sim Cn^{1/2} n^n e^{-n}. \tag{2}$$

2. The second method. In the second method we consider $\log n!$ and represent it as a sum:

$$\log n! = \sum_{k=1}^{n} \log k. \tag{9}$$

It is quite natural to try to approximate the right hand side of (9) by $\int_1^n \log t\, dt$, and in fact it is true that

$$\log n! = \int_1^n \log t\, dt + O(1) = n \log n - n + O(1). \tag{10}$$

However, (10) is not equivalent to (2) although (2) does imply (10). A proposition equivalent to (2), involving $\log n!$ is

$$\log n! = (n + \tfrac{1}{2}) \log n - n + \log C + o(1). \tag{11}$$

From the monotonicity of $\log x$ it follows that

$$\int_1^n \log [t]dt \leqq \int_1^n \log t\, dt \leqq \int_1^n \log [t + 1]dt \tag{12}$$

where $[x]$ denotes the greatest integer in x. That is,

$$\log (n - 1)! \leqq \int_1^n \log t\, dt \leqq \log n!. \tag{13}$$

Thus monotonicity yields only

$$\log n! = \int_1^n \log t\, dt + O(\log n). \tag{14}$$

In order to improve (14) we use our second lemma which takes advantage of the concavity of $\log x$.

LEMMA II. *Let $f(x)$ be monotone increasing and concave for $x>a$. Then*

$$0 \leqq \sum_{k>a} \left\{ \int_k^{k+1} f(t)dt - \frac{f(k) + f(k+1)}{2} \right\} \leqq \frac{f(a+1) - f(a)}{2}.$$

Proof. The first inequality follows from the concavity of $f(x)$, since each term of the sum is non-negative. It expresses the fact that the graph of a concave function lies above the chord.

The second inequality states that the sum of the areas between the function and its chords is a convergent series if the end-points of the chords have abscissae at the integers.

To show this, we consider the point set I_n bounded by the graph of $f(x)$ and the chord joining the points $(n, f(n))$ and $(n+1, f(n+1))$ for $n>a$ and translate it (without rotation) so that the point $(n+1, f(n+1))$ becomes the point $(m+1, f(m+1))$ where $m=[a+1]$.

It follows from the concavity of $f(x)$, that after translation the interior of I_n will not intersect the interior of I_{n+1} and will lie completely below I_{n+1}. From the monotonicity of $f(x)$ it follows that for each n, I_n will lie between the horizontal lines $y=f(m)$, $y=f(m+1)$.

Hence, all these areas can fit in the triangle whose vertices are $(m, f(m))$, $(m, f(m+1))$, $(m+1, f(m+1))$ without overlapping (their boundaries may intersect), and hence if A_n denotes the area of I_n,

$$\sum_{n>a} A_n \leqq \frac{f(m+1)-f(m)}{2}.$$

But by the concavity of $f(x)$,

$$f(m+1)-f(m) \leqq f(a+1)-f(a).$$

Since

$$A_k = \int_k^{k+1} f(t)\,dt - \frac{f(k)+f(k+1)}{2},$$

we thus obtain Lemma II.

An alternative proof can be based on the inequality which we state here without proof:

$$A_n \leqq \tfrac{1}{2}\{f(n+1)-f(n)\} - \tfrac{1}{2}\{f(n+2)-f(n+1)\}.$$

We now apply Lemma II to our problem: Let $a=1$, $f(x)=\log x$. Then:

$$0 \leqq \sum_{k=2}^{\infty} \left\{\int_k^{k+1} \log t\,dt - \frac{\log k + \log (k+1)}{2}\right\} \leqq \tfrac{1}{2}\log 2.$$

Since the terms of the series are positive, we have

$$\lim_{N\to\infty} \sum_{k=2}^{N} \left\{\int_k^{k+1} \log t\,dt - \frac{\log k + \log (k+1)}{2}\right\} = D \leqq \tfrac{1}{2}\log 2,$$

or

$$\lim_{N\to\infty} \left\{\int_2^{N} \log t\,dt - \frac{\log (N-1)! + \log N!}{2}\right\} = D - \tfrac{1}{2}\log 2.$$

Since $\log N! = \log (N-1)! + \log N$, we get

$$\lim_{N\to\infty} \{N\log N - N - 2\log 2 + 2 - \log N! + \tfrac{1}{2}\log N\} = D - \tfrac{1}{2}\log 2.$$

$$\therefore \log N! = (N+\tfrac{1}{2})\log N - N - D_N,$$

where

$$\lim_{N\to\infty} D_N = D + \frac{3}{2}\log 2 - 2 = \log C.$$

Hence

$$\log N! = (N+\tfrac{1}{2})\log N - N + \log C + o(1),$$

and the proof is complete.

MATHEMATICAL NOTES

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NOTES ON THE SKEW QUADRILATERAL

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In the February 1953 issue of this journal Thébault* has established two interesting theorems on the skew quadrilateral. We give some supplementary remarks on both of them.

A. The first states: *a skew quadrilateral which has its opposite sides equal in pairs has its opposite angles equal in pairs and conversely.* The second part of the theorem is not so very simple; the geometrical proof given by the author is elegant, but requires some confidence in the possibility of a certain construction. A proof can also be given along algebraic lines. $ABCD$ be the skew quadrilateral, $AB=a$, $BC=b$, $CD=c$, $DA=d$, $(AD, AB)=\alpha$, $(BA, BC)=\beta$, $(CB, CD)=\gamma$, $(DC, DA)=\delta$. Given $\alpha=\gamma$, $\beta=\delta$. If $AC=p$, $BD=q$, we have in the triangles ABC and CDA respectively

$$p^2 = a^2 + b^2 - 2ab\cos\beta, \qquad p^2 = c^2 + d^2 - 2cd\cos\delta;$$

hence $\beta=\delta$ implies

$$\frac{a^2+b^2-p^2}{ab} = \frac{c^2+d^2-p^2}{cd}$$

or

$$(ab - cd)p^2 = -(ad - bc)(ac - bd). \tag{1}$$

Similarly

$$(ad - bc)q^2 = -(ab - cd)(ac - bd). \tag{2}$$

Now we have two cases: $ab-cd\neq 0$, and $ab-cd=0$.

I. If $ab-cd\neq 0$, then in view of (1) $ad-bc\neq 0$, $ac-bd\neq 0$. Thus

$$p^2 = -\frac{(ad-bc)(ac-bd)}{ab-cd}, \qquad q^2 = -\frac{(ab-cd)(ac-bd)}{ad-bc},$$

and therefore

$$p^2q^2 = (ac - bd)^2;$$

that means

$$ac = bd + pq \quad \text{or} \quad bd = ac + pq.$$

* V. Thébault, On the skew quadrilateral. This MONTHLY, vol. 60, 1953, pp. 102–105.

In both cases by a well-known theorem§ the four points A, B, C, and D are coplanar, so that the quadrilateral is not a skew one.

II. If $ab-cd=0$, then according to (2) $ad-bc=0$ and we have indeed $a=c$, $b=d$.

B. The second of Thébault's theorems says: *a skew quadrilateral has its opposite sides equal in pairs if and only if the perpendiculars at the vertices to the planes of the two sides meeting at these vertices constitute a hyperbolic group of lines.* He gives a short and interesting proof of both statements.

If $ABCD$ is the quadrilateral, $AB=a$, *etc.*, h_A the perpendicular at A to the plane DAB, *etc.*, then the theorem stating that $a=c$, $b=d$ implies that h_A, h_B, h_C, h_D, are on a hyperboloid was to the best of my knowledge first given by Bennett* in his study of the "isogram," which he considered above all from a kinematical standpoint. The converse theorem was given by me as a "problem" in a Dutch mathematical paper† in 1950; several solutions were sent in, and some of them seem as simple and straightforward as Thébault's. There would not be any reason to return to the matter, but for the fact that Bennett's proof of his theorem shows an additional property of the said hyperboloid. Therefore we reproduce it here in a few words.

For the quadrilateral, let $a+b=c+d$. Produce AB with $BP=b$ and AD with $DQ=c$, then $AP=AQ$. The outer bisecting plane W_B of the angle ABC intersects PC orthogonally in its midpoint; so does W_D the line QC. The inner bisecting plane V_A intersects PQ orthogonally in its midpoint. Hence V_A, W_B and W_D have a line r_1 in common and evidently the inner bisecting plane V_C passes through this line too. Now if $a=c$, $b=d$ we have not only $a+b=c+d$, but $a+d=c+b$ as well. That means: V_B, V_D, W_A and W_C pass through a line r_2. We fix a projectivity between the planes of the pencil (r_1) and those of the pencil (r_2) by associating with a plane of (r_1) that one of (r_2) which is perpendicular to it. Now it is a well-known theorem in projective geometry that the locus of the lines of intersection of corresponding planes in two projective pencils is a regulus. As V_A and W_A are perpendicular planes meeting in h_A, *etc.*, we conclude that h_A, h_B, h_C, h_D constitute a hyperbolic group of lines. But according to another well-known theorem of projective geometry, in view of the special manner in which the projectivity of the pencils is defined here, we have the additional property: the hyperboloid to which the four perpendiculars h_A, h_B, h_C and h_D belong is orthogonal.

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§ A proof of Ptolemy's theorem in space can be given as follows: Use an inversion with its centre in A, transforming B, C and D into B_1, C_1, D_1. Then

$$B_1C_1 = \frac{k \cdot BC}{AB \cdot AC} = \frac{k \cdot b}{ap},\ C_1D_1 = \frac{kc}{pd},\ D_1B_1 = \frac{kq}{da}.$$

Hence $C_1D_1 = C_1B_1 + B_1D_1$ or $B_1C_1 = B_1D_1 + D_1C_1$; that means that B_1, C_1 and D_1 are collinear. Consequently A, B, C and D are coplanar (and of course on a circle).

* Proceedings of the London Math. Soc. vol. 13, 1914, p. 151.

† Wiskundige Opgaven met de oplossingen, dl. 19, 2, (1951) problem XLI, pp. 83–84.

THREE DIMENSIONAL HARMONIC FUNCTIONS GENERATED BY ANALYTIC FUNCTIONS OF A HYPERVARIABLE

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1. Introduction. The study of harmonic functions in two dimensions is greatly simplified by the theorem that

$$V = f(x + iy) + g(x - iy) \tag{1}$$

is the general solution of $V_{xx}+V_{yy}=0$ where f and g are analytic functions. The analogous result in three dimensions [4] gives the general solution of $V_{xx}+V_{yy}+V_{zz}=0$ as an integral of the form

$$V = \int_{-\pi}^{\pi} f(z + ix\cos u + iy\sin u, u)du \tag{2}$$

from which the extraction of solutions in more explicit form is much more difficult than in the two dimensional case.

Von Beckh-Widmanstetter [1] has shown that the general solution of Laplace's equation in three dimensions can not be generated by a hypervariable with analytic functions having only three components. Various authors have constructed special solutions with the aid of hypervariables, and Ketchum [2] has obtained the general solution by using a hypervariable with analytic functions having infinitely many components. The author forms particular solutions from the components of $(\omega-1)f(t)$ where $t=x+\omega y+\omega^2 z$, $\omega^3=1$ and f is an analytic function of t. Since only two of the three component functions are linearly independent and the functions are constant on any line normal to the plane $x+y+z=0$, the harmonic functions are in essence two dimensional, and the author closes with the transformation relating them to the ordinary components of an analytic function of a complex variable.

2. Properties of analytic functions of t.

DEFINITION. *$f(t)=(u, v, z)+\omega v(x, y, z)+\omega^2 w(x, y, z)$ is said to be analytic at $t=t_0$ if $\lim_{\Delta t\to 0}[f(t+\Delta t)-f(t)]/\Delta t$ exists in a neighborhood of t_0.* The generalized Cauchy Riemann Equations

$$u_x = v_y = w_z, \quad u_y = v_z = w_x \quad \text{and} \quad u_z = v_x = w_y \tag{3}$$

obviously hold throughout a neighborhood of t_0 if $f(t)$ is analytic at t_0. If u, v and $w\in C^2$ at t_0 we have from equation (3) and the equality of cross derivatives:

$$\nabla^2 u = \nabla^2 v = \nabla^2 w \tag{4}$$

from which we conclude

THEOREM 1. *The function,*

$$F(t) = (\omega - 1)f(t) = U + \omega V + \omega^2 W \tag{5}$$

has components $U=w-u$, $V=u-v$ and $W=v-w$ which are harmonic and satisfy

the equation $U+V+W=0$.

The usual laws establishing the analyticity and derivatives of (a) analytic functions of an analytic function, (b) sums and products of an analytic function, (c) positive and negative integral powers of t and (d) all functions which can be built up by rational operations on t follow as in Ward's paper [3]. Also, we may generalize the exponential and trigonometric functions from their usual series expansion and obtain the usual formulas for their derivatives.

3. Examples of harmonic functions generated by Theorem 1.

(a) The components of $(\omega-1)t^3$ are readily seen to be

$$U = -x^3 + 3x^2z + 3xy^2 - 6xyz - y^3 + 3yz^2 - z^3$$
$$V = x^3 - 3x^2y + 6xyz - 3xz^2 + y^3 - 3y^2z + z^3$$
$$W = 3x^2y - 3x^2z - 3xy^2 + 3xz^2 - 3yz^2 + 3y^2z.$$

It is to be noted that the components of $(\omega-1)t^n$ for n integral provide only two independent homogeneous harmonic polynomials of degree n whereas the general solution in series form involves linear combinations of $2n+1$ such polynomials [4] for each degree n.

(b) Consider $F(t)=(\omega-1)e^t=(\omega-1)e^xe^{\omega y}e^{\omega^2 z}$. Let

$$(6)\quad g_1(s) = 1/3\left[e^s - e^{-\frac{1}{2}s}\cos\frac{\sqrt{3}}{2}s - \sqrt{3}\,e^{-\frac{1}{2}s}\sin\frac{\sqrt{3}}{2}s\right] = \sum_{n=0}^{\infty}\frac{s^{3n+2}}{(3n+2)!}$$

$$(7)\quad g_2(s) = g_1'(s) = \sum_{n=0}^{\infty}\frac{s^{3n+1}}{(3n+1)!}$$

$$(8)\quad g_3(s) = g_1''(s) = \sum_{n=0}^{\infty}\frac{s^{3n}}{(3n)!};$$

then

$$F(t) = (\omega-1)e^x[g_3(y)+\omega g_2(y)+\omega^2 g_1(y)][g_3(z)+\omega g_1(z)+\omega^2 g_2(z)]$$
$$= U+\omega V+\omega^2 W,$$

where

$$(9)\quad U = e^x\{g_1(y)[g_3(z)-g_1(z)]+g_2(y)[g_1(z)-g_2(z)]+g_3(y)[g_2(z)-g_3(z)]\}$$
$$(10)\quad V = e^x\{g_1(y)[g_1(z)-g_2(z)]+g_2(y)[g_2(z)-g_3(z)]+g_3(y)[g_3(z)-g_1(z)]\}$$
$$(11)\quad W = e^x\{g_1(y)[g_2(z)-g_3(z)]+g_2(y)[g_3(z)-g_1(z)]+g_3(y)[g_1(z)-g_2(z)]\}.$$

With the aid of the recursion formulas for $g_k(s)$, $k=1, 2, 3$, we can readily show that U is harmonic as we know it must be from Theorem 1. Since $U+V+W=0$, $U_{xx}=U$, $U_{yy}=V$ and $U_{zz}=W$, we have immediately

$$(12)\quad \nabla^2 U = U+V+W = 0,$$

or expanding (9), (10) and (11) by means of (6), (7) and (8) we have

$$U = \frac{2}{\sqrt{3}} e^{x-\frac{1}{2}y-\frac{1}{2}z} \sin\left[\frac{\sqrt{3}}{2}(y-z) + \frac{4\pi}{3}\right] = U_{zz}, \tag{13}$$

$$V = \frac{2}{\sqrt{3}} e^{x-\frac{1}{2}y-\frac{1}{2}z} \sin\left[\frac{\sqrt{3}}{2}(y-z) + \frac{2\pi}{3}\right] = U_{yy}, \tag{14}$$

$$W = \frac{2}{\sqrt{3}} e^{x-\frac{1}{2}y-\frac{1}{2}z} \sin\left[\frac{\sqrt{3}}{2}(y-z)\right] = U_{xx}, \tag{15}$$

from which (12) also follows by addition.

4. Relation to two dimensional harmonic functions. The laws for differentiating elementary analytic functions of t correspond in a one-one fashion with those for differentiating the corresponding analytic functions of a complex variable $Z=X+iY$. In fact, we note that when

$$f(t) = \sum_{n=0} a_n t^n,$$

where the summation is finite or infinite, the computation of the expression $(\omega-1)[f(t+\Delta t)-f(t)]$ in the special case $\Delta t=\Delta x+\omega\Delta y+\omega^2\Delta z$, with $\Delta x=\Delta y=\Delta z$, leads to

$$\begin{aligned}(\omega-1)&[f(t+\Delta t)-f(t)] \\ &= (\omega-1)\sum_{n=1} a_n\Delta t\left[(t+\Delta t)^{n-1} + (t+\Delta t)^{n-2}t + \cdots + t^{n-1}\right] = 0\end{aligned} \tag{16}$$

since $(\omega-1)\Delta t=0$ in this case. Thus the functions (5) are constant on any line normal to the plane $x+y+z=0$. Since they are harmonic in 3 dimensions and have directional derivatives zero normal to that plane, the plane section of the functions (5) in $x+y+z=0$ must be two dimensional harmonic within that plane and hence are associated with analytic functions of a complex variable in that plane.

Heretofore, only the property $\omega^3=1$ has been used in forming functions $(\omega-1)f(t)$ with harmonic components. The complex number $\omega=\frac{1}{2}(-1+i\sqrt{3})$ has the additional property that $\omega^2+\omega+1=0$, which destroys uniqueness in the expansion of $f(t)$. However, functions equivalent modulo $\omega^2+\omega+1$ to the functions (5) have harmonic components which are linear combinations of those given by Theorem 1. The decomposition into real and imaginary parts of

$$Z = X + iY = t = x + \omega y + \omega^2 z = x - \tfrac{1}{2}y - \tfrac{1}{2}z + i\frac{\sqrt{3}}{2}(y-z) \tag{17}$$

is obtained by setting $\omega=\frac{1}{2}(-1+i\sqrt{3})$. Letting $f(Z)$ be an analytic function of Z, and using the relation $i=(\omega-\omega^2)/\sqrt{3}$, we have

$$
\begin{aligned}
(\omega - 1)f(Z) &= (\omega - 1)[U(X, Y) + iV(X, Y)] \\
&= (\omega - 1)\left[U(X, Y) + \omega \frac{V(X, Y)}{\sqrt{3}} - \omega^2 \frac{V(X, Y)}{\sqrt{3}}\right] \\
&= \left[-\frac{V(X, Y)}{\sqrt{3}} - U(X, Y)\right] + \omega\left[U(X, Y) - \frac{V(X, Y)}{\sqrt{3}}\right] \\
&\quad + \omega^2\left[\frac{2V(X, Y)}{\sqrt{3}}\right] \\
&= U^*(X, Y) + \omega V^*(X, Y) + \omega^2 W^*(X, Y),
\end{aligned} \tag{18}
$$

where, by (17), we may write

$$
\begin{aligned}
U^*(X, Y) = -\Big[\frac{1}{\sqrt{3}} V\Big(x - \tfrac{1}{2}y - \tfrac{1}{2}z, \frac{\sqrt{3}}{2}(y - z)\Big) \\
+ U\Big(x - \tfrac{1}{2}y - \tfrac{1}{2}z, \frac{\sqrt{3}}{2}(y - z)\Big)\Big]
\end{aligned} \tag{19}
$$

and corresponding formulas for V^* and W^*. Since the components of e^Z are $U_1 = e^X \cos Y$ and $V_1 = e^X \sin Y$, the expansion of U in (13) follows readily from (19). The results of (18), (19) and Theorem 1 may be combined to prove the closing

THEOREM 2. *Let $f_1(Z) = U_1(X, Y) + iV_1(X, Y)$ be an analytic function of $Z = X + iY$. The functions*

$$
U_1\Big(x - \tfrac{1}{2}y - \tfrac{1}{2}z, \frac{\sqrt{3}}{2}(y - z)\Big) \quad \textit{and} \quad V_1\Big(x - \tfrac{1}{2}y - \tfrac{1}{2}z, \frac{\sqrt{3}}{2}(y - z)\Big)
$$

are harmonic functions related to those of Theorem 1 *by the equations*

$$
U = -V_1/\sqrt{3} - U_1, \quad V = U_1 - V_1/\sqrt{3} \quad \textit{and} \quad W = 2V_1/\sqrt{3}.
$$

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NOTE CONCERNING A METHOD FOR FINDING PRIMES

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In his paper [1] Thompson notes that if the first n primes are split into two different groups

$$q_1, q_2, \cdots, q_a; \qquad r_1, r_2, \cdots, r_b,$$

and if the difference between the two products

$$D = (q_1 q_2 \cdots q_a) - (r_1 r_2 \cdots r_b) = Q - R \tag{1}$$

should turn out to be less than $(p_n+2)^2$ (it should be added: in absolute value and greater than 1), then D is a prime. A similar result is also given in [2]. Thompson asks whether it is always possible by a suitable choice of the products in (1) to obtain

$$|D| < (p_n + 2)^2. \tag{2}$$

It is shown here that the answer is negative. In fact, we shall show that for $n=10$, no matter how the q's and r's are chosen

$$|D| \geqq 1811, \tag{3}$$

and that $|D|=1811$ occurs only for

$$\begin{aligned} Q &= Q_1 = 2\cdot 7\cdot 13\cdot 19\cdot 23 = 79534 \quad \text{and} \\ R &= R_1 = 3\cdot 5\cdot 11\cdot 17\cdot 29 = 81345, \end{aligned} \tag{4}$$

while $(p_{10}+2)^2=961$.

We shall have proved (3) if we can show that for any choice of the q's and r's other than the one given in (4) either $Q<Q_1$ or $Q>R_1$.

Without loss of generality we can suppose $q_1<q_2<q_3<\cdots<q_a$. We first observe that for (2) to hold in the case $n=10$ we need $a=b=5$. Suppose $a=7$, then the maximum value of Q is $Q=19\cdot 23\cdot 29=6673$ and $Q<Q_1$. Suppose $a=6$, then for $q_1\cdot q_2\cdot q_3\cdot q_4\cdot q_5=2\cdot 3\cdot 5\cdot 7\cdot 11$, *i.e.*, the product of the first 5 primes and q_6 the largest possible value, *i.e.* 29, the value of Q obtained is less than Q_1; continuing to consider systematically the other cases for which $a=6$, we find that $Q<Q_1$ or $Q>R_1$, which shows that a cannot equal 6.

Considering now the case $a=b=5$, we can suppose without loss of generality that $q_1=2$. By a series of a few inequalities in each of which $2=q_1<q_2<q_3<q_4<q_5$, the reader can readily verify that except for the choice of q's given in (4) either $Q<Q_1$ or $Q>R_1$.

It is to be noted that this does not imply that it may not be possible to produce a prime for D (indeed $D=1811$ is a prime), but it cannot always be done by (2).

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GEOMETRY OF THE TETRAHEDRON*

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This note has as its object a new or little known theorem which may be used to find properties of tetrahedrons in which the Monge point is situated in the plane of one face. [1].

THEOREM. *If the plane of three points a, b, c, situated respectively in the planes of the faces DBC, DCA, DAB of a tetrahedron $T \equiv ABCD$, passes through the vertex D, the plane determined by the isogonal conjugates a', b', c' of a, b, c with respect to the corresponding faces also passes through the point D, and conversely.*

Proof. Let the segments $DA' = DB' = DC'$ be placed on the edges DA, DB, DC. The lines Da and Da', Db and Db', Dc and Dc' meet $B'C'$, $C'A'$, $A'B'$ in a_1 and a_1', b_1 and b_1', c_1 and c_1'. The bisectors of the angles at D of the faces DBC, DCA, DAB are perpendicular to the sides of the triangle $A'B'C'$ at their midpoints. It follows that the points a_1, b_1, c_1, situated in the plane $Dabc$, are collinear and symmetric to the points a_1', b_1', c_1' with respect to the midpoints of $B'C'$, $C'A'$, $A'B'$. In other words, the points a_1', b_1', c_1' are on the reciprocal transversal d_1' of the line $d_1 \equiv a_1b_1c_1$ with respect to the triangle $A'B'C'$. Hence the plane $a'b'c'$ is the plane (D, d_1') and conversely since the sets of points (a, b, c) and (a', b', c') may be interchanged.

COROLLARY. *If the Monge point of a tetrahedron T is situated in the plane of one face, the opposite vertex and the orthocenters of the adjacent faces are coplanar and conversely. The opposite vertex and the circumcenters of the adjacent faces are also coplanar and conversely.*

Proof. If the Monge point is in the face ABC, then the foot D' of the altitude DD' of T is on the circumcircle of ABC [1] and its orthogonal projections A_1, B_1, C_1 on the edges BC, CA, AB (*i.e.*, the feet of the altitudes DA_1, DB_1, DC_1 of faces DBC, DCA, DAB) are on the Simson line S_1 of D' with respect to triangle ABC. Hence the plane (D, S_1) contains the altitudes DA_1, DB_1, DC_1 and consequently the orthocenters of the faces DBC, DCA, DAB. Conversely, if the plane of the orthocenters of the faces DBC, DCA, DAB passes through the vertex D, it contains the points A_1, B_1, C_1. The perpendiculars to BC, CA, AB drawn through A_1, B_1, C_1 intersect at the foot D' of the altitude DD'. Hence D' must be on the circumcircle of ABC as A_1, B_1, C_1 are collinear by hypothesis. As D' is on the sphere $ABCD$, the Monge point of T is in the plane ABC. Furthermore, according to the preceding theorem the vertex D of T and the circumcenters of DBC, DCA, DAB situated on the isogonals of the altitudes DA_1, DB_1, DC_1 with respect to the corresponding faces are likewise coplanar and conversely.

COROLLARY. *If the Monge point is in the plane of face ABC of T, the isotomic*

* Translated by W. E. Byrne.

conjugates a'', b'', c'' of the orthocenters of faces DBC, DCA, DAB with respect to the corresponding triangles are in a plane passing through the vertex D and conversely.

Proof. Since the points A_1, B_1, C_1 are on the Simson line S_1 of D' with respect to triangle ABC, the points of intersection A_2, B_2, C_2 of the isotomic conjugates of DA_1, DB_1, DC_1 in faces DBC, DCA, DAB, are on the reciprocal transversal S_2 of S_1 with respect to triangle ABC. S_2 coincides with the Simson line with respect to triangle ABC of the point D'' diametrically opposite D and D' on the sphere $ABCD$ and on the circle ABC. Hence the points a'', b'', c'' on DA_2, DB_2, DC_2 are coplanar with D and conversely. Note that the points a'', b'', c'' coincide in each of the triangles DBC, DCA, DAB with the anti-complementary point of the symmedian point. [2].

COROLLARY. *The planes ABC, (D, S_1), (D, S_2) meet on the nine-point circle of the triangle ABC in the orthopole of diameter $D'D''$ of the circle ABC.*

Proof. According to a known theorem [3] the Simson lines S_1 and S_2 are perpendicular and they intersect at the orthopole of the diameter $D'D''$ of circle ABC. The orthopole in question is on the nine-point circle of triangle ABC. [4]

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ON A FORMULA OF GROSSWALD*

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Using known properties of the Legendre polynomials $P_n(x)$ and the hypergeometric function $F(a, b; c; x)$, Grosswald [2] has recently derived the formula

$$(1)\qquad \sum_{k=0}^{n-r}(-2)^{-k}\binom{n}{r+k}\binom{n+r+k}{k} = \begin{cases}(-1)^{(n-r)/2}2^{-(n-r)}\binom{n}{(n-r)/2}, & r \equiv n \pmod 2\\ 0, & r \not\equiv n \pmod 2\end{cases}$$

and has remarked that (1) does not follow readily by the methods of [3] and that a direct proof seems rather difficult. Carlitz [1] has shown that the known formula [Gauss],

* *Editorial note.* Within a month after this paper had been accepted for publication, a note containing very similar results was submitted by B. S. Popov, University of Skopje, Yugoslavia.

$$(2) \qquad F(a, b; \tfrac{1}{2}(a+b+1); \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)}$$

yields (1) as a special case for $a=-(n-r)$, $b=n+r+1$ .We wish to point out that (1) can actually be recovered from the formula (see [3], p. 253):

$$(3) \qquad \sum_{k=0}^{n} (-2)^k \binom{\mu}{k}\binom{2\mu-2k}{n-k} = \begin{cases} \binom{\mu}{n/2}, & n = 0, 2, \cdots \\ 0, & n = 1, 3, \cdots, \end{cases}$$

which is the result of equating the coefficients of z^n on both sides of the identity:

$$(1+z)^{2\mu}(1-2z(1+z)^{-2})^{\mu} = (1+z^2)^{\mu}.$$

Indeed, following Grosswald, one is naturally led to (3) by analogous considerations of the ultraspherical polynomials $P_n^{\lambda}(x)$, for which we note the explicit representation (see [4], p. 84):

$$(3a) \qquad P_n^{\lambda}(x) = \sum_{m=0}^{[n/2]} (-1)^m \binom{\lambda+n-m-1}{n-m}\binom{n-m}{m}(2x)^{n-2m},$$

and the hypergeometric representation (see [4], p. 80):

$$(3b) \qquad P_n^{\lambda}(x) = \binom{2\lambda+n-1}{n} F(-n, 2\lambda+n; \lambda+\tfrac{1}{2}; (1-x)/2).$$

Thus one finds (3), without difficulty, by equating the values of $P_n^{\lambda}(0)$ derived from (3a) and (3b) and setting $\lambda=-\mu$. One may also verify, after Carlitz, that (2) yields (3) as a special case for $a=-n$, $b=-2\mu+n$.

Now taking into account the relation (see [4], p. 83):

$$P_n^{(r)}(x) = 1\cdot 3\cdot\cdots\cdot(2r-1)P_{n-r}^{r+1/2}(x)$$

in which $P_n^{(r)}(x)$ is the rth derivative of $P_n(x)\equiv P_n^{1/2}(x)$ and recalling Grosswald's derivation of (1), one certainly recovers (1) from (3) by taking $n-r$ in place of n and setting $-\mu=r+\frac{1}{2}$ in (3).

That (1) is a special case of (3) as just indicated is by no means apparent without the above facts in the background. It is therefore desirable to point out a simple and direct proof of (1). To this end, we just note that, on changing the summation index k in (1) into $(n-r)-k$, (1) appears at once as a special case of the formula:

$$(4) \qquad \sum_{k=0}^{n} (-2)^k \binom{\mu}{k}\binom{2\mu-k}{n-k} = \begin{cases} (-1)^{n/2}\binom{\mu}{n/2}, & n = 0, 2, \cdots \\ 0, & n = 1, 3, \cdots, \end{cases}$$

which is the result of equating the coefficients of z^n on both sides of the identity:

$$(1+z)^{2\mu}(1-2z(1+z)^{-1})^{\mu}=(1-z^2)^{\mu}.$$

In fact, we regain (1) from (4) by taking $n-r$ in place of n and then $\mu=n$ in (4).

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THE CHARACTERISTIC INITIAL VALUE PROBLEM FOR THE WAVE EQUATION AND RIEMANN'S METHOD

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1. Introduction. Riemann's method has been used extensively for solving the Cauchy problem for hyperbolic equations in two independent variables. This method has been extended to the wave equation in more than two variables by Lewy [4], Martin [5], and Diaz and Martin [3]. The determination of the Riemann function in these cases hinges on the solution of a characteristic initial value problem. The ability to do this (usually by iterations for two independent variables) is presupposed, or in some cases the Riemann function can be found explicitly.

If a solution of a characteristic initial value problem is sought by Riemann's method then the natural way of determining the Riemann function leads to another characteristic initial value problem and the process becomes circular. Instead of the usual procedure, a variation, first employed by Martin for the Cauchy problem [5], will be used to overcome this difficulty for the case of certain hyperbolic equations in more than two independent variables. The argument will be carried through for the wave equation in three variables

$$u_{xx}+u_{yy}=u_{zz} \tag{1}$$

although it will be clear from the discussion that the result holds for hyperbolic equations with constant coefficients in any number ($\geqq 3$) of variables. In addition certain cases where the coefficients depend only on z can be handled without additional difficulty.

2. The characteristic initial value problem. Without loss of generality we consider the characteristic cone

$$x^2+y^2=(z-1)^2 \tag{2}$$

of equation (1) and seek a solution of this equation interior to (2), $0\leqq z\leqq 1$, satisfying the condition

$$u(x, y, 1-\sqrt{x^2+y^2})=\psi(x, y), \qquad 0\leqq x^2+y^2\leqq 1. \tag{3}$$

The given function $\psi(x, y)$ is assumed to have continuous second derivatives. We transform (1) to characteristic coordinates by introducing (α, β, ϕ) as new variables according to the relations

$$x = \tfrac{1}{2}(\alpha - \beta)\cos\phi, \qquad y = \tfrac{1}{2}(\alpha - \beta)\sin\phi, \qquad z = \tfrac{1}{2}(\alpha + \beta).$$

Then equation (1) becomes

$$Lu \equiv u_{\alpha\beta} - \frac{1}{2(\alpha - \beta)}(u_\alpha - u_\beta) - \frac{u_{\phi\phi}}{(\alpha - \beta)^2} = 0. \tag{4}$$

It is easy to see that the surfaces $\alpha = \text{const.}$, $\beta = \text{const.}$ are characteristic cones with vertices on the z-axis, while $\phi = \text{const.}$ represents the half-planes containing the z-axis. Following this transformation we introduce the *associate equation* to (4):

$$Mv \equiv v_{\alpha\beta} + \frac{1}{2(\alpha - \beta)}(v_\alpha - v_\beta) = 0.$$

This equation assumes the role of the *adjoint equation* usually considered in such problems, but is essentially different in that it depends only on the variables α and β. This dependence on two variables (the characteristic variables) continues to hold for hyperbolic equations (with constant coefficients) in any number of variables.

If D is any domain in (α, β, ϕ) space with boundary B we may easily verify the identity

$$\begin{aligned}\iiint_D [(v_\beta - v_\alpha)Lu - (u_\beta - u_\alpha)Mv]\,d\alpha d\beta d\phi \\ = \iint_B \left[u_\beta v_\beta d\beta d\phi - u_\alpha v_\alpha d\alpha d\phi + u_\phi \frac{v_\alpha - v_\beta}{(\alpha - \beta)^2}\, d\alpha d\beta\right].\end{aligned} \tag{5}$$

Let $P(0, 0, z_0)$, $0 \leqq z_0 \leqq 1$, be a point at which we wish to determine the solution. Once the solution is obtained at points on the z-axis we may employ a Lorentz transformation to obtain the solution at all interior points of the cone (see [2, p. 442]).

In (x, y, z) space we consider a domain D^* bounded by the portions of the two cones

$$x^2 + y^2 = (z - 1)^2, \qquad \tfrac{1}{2}(1 + z_0) \leqq z \leqq 1;$$
$$x^2 + y^2 = (z - z_0)^2, \qquad z_0 \leqq z \leqq \tfrac{1}{2}(1 + z_0).$$

If (α, β, ϕ) is considered as a system of rectangular coordinates then the domain D^* when transformed to (α, β, ϕ) space is the (infinite) triangular prism bounded by the planes $\alpha = 1$, $\beta = z_0$, $\alpha = \beta$. Since we are interested only in solutions of (4) which have period 2π we limit our considerations to the section of the prism

contained between the planes $\phi=0$ and $\phi=2\pi$. We designate this domain as D and its boundary B.

If u is the desired solution of $Lu=0$ and v is a regular solution of $Mv=0$ yet to be prescribed then (5) will vanish. Since v is independent of ϕ those integrals on the right side of (5) taken over the part of B lying in $\phi=0$ and $\phi=2\pi$ will cancel. We therefore have

$$\iint [u_\beta v_\beta]_{\alpha=1} d\beta d\phi - \iint [u_\alpha v_\alpha]_{\beta=z_0} d\alpha d\phi + \iint [u_\beta v_\beta - u_\alpha v_\alpha]_{\alpha=\beta} d\alpha d\phi = 0. \tag{6}$$

We now seek a solution of $Mv=0$ which has the property that

$$v_\alpha = -1 \text{ on } \beta = z_0 \quad \text{and} \quad v_\alpha = v_\beta = 0 \text{ on } \alpha = \beta.$$

This is a problem of Goursat type for a hyperbolic equation in two independent variables and any such solution will be termed a Riemann function. It is easy to verify that the solution

$$v = -(\alpha + \beta) + 2(\alpha - z_0)^{1/2}(\beta - z_0)^{1/2}$$

satisfies the required conditions. Inserting this value for v in the above equation (6) we obtain

$$\int_0^{2\pi} \int_{z_0}^1 [u_\alpha]_{\beta=z_0} d\alpha d\phi = \int_0^{2\pi} \int_{z_0}^1 [u_\beta]_{\alpha=1} \left[\left(\frac{1-z_0}{\beta - z_0} \right)^{1/2} - 1 \right] d\beta d\phi$$

or

$$\frac{1}{2\pi} \int_0^{2\pi} [u]_{\alpha=1,\beta=z_0} d\phi - \frac{1}{2\pi} \int_0^{2\pi} [u]_{\beta=z_0,\alpha=z_0} d\phi = \frac{1}{2\pi} \int_0^{2\pi} \int_{z_0}^1 [u_\beta]_{\alpha=1} \left[\left(\frac{1-z_0}{\beta - z_0} \right)^{1/2} - 1 \right] d\beta d\phi.$$

When transformed to (x, y, z) coordinates the second integral on the left above is seen to be the value of u at the point P. In the integral on the right u is known $(=\psi)$ in the plane $\alpha=1$ and since

$$\frac{\partial}{\partial \beta} = -\frac{1}{2} \left(\frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right)$$

is an inner derivative (*i.e.*, derivative in a characteristic direction) we know the value of u_β in this plane. Hence we finally find

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} [\psi]_{\alpha=1,\beta=z_0} d\phi - \frac{1}{2\pi} \int_0^{2\pi} [\psi_\beta]_{\alpha=1} \left[\left(\frac{1-z_0}{\beta - z_0} \right)^{1/2} - 1 \right] d\beta d\phi. \tag{7}$$

The fact that u given by (7) on the z-axis and by Lorentz transformations of (7) at other points actually satisfies the differential equation may be established by differentiation. The justification of the differentiations of the various integrals involved has been given by d'Adhémar [1] in the process of obtaining a solution to the same problem considered here. Equation (7) gives an explicit formula for the solution of this problem and the method seems much simpler than that given in [2] which depends on a mean value theorem of Asgeirsson and the solution of an Abel integral equation.

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CLASSROOM NOTES

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ON THE VECTOR TRIPLE PRODUCT

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In this note, different methods of expanding the vector triple product will be reviewed. In addition, two new variations will be presented.

A. A direct method [1] is to expand $\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$ and $[(\mathbf{N}\cdot\mathbf{C})\mathbf{B}-(\mathbf{N}\cdot\mathbf{B})\mathbf{C}]$ into their $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components and compare. Although this method is direct it is inelegant.

B. Another direct method [2] but with little calculation is to use the permutation symbol ϵ_{rst}. Thus for the rth component of $\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$ we have

$$(1)\quad [\mathbf{N}\times(\mathbf{B}\times\mathbf{C})]_r = \epsilon_{rst}\mathbf{N}_s[\epsilon_{tuv}\mathbf{B}_u\mathbf{C}_v] = \mathbf{N}_s\delta_{rs}^{uv}\mathbf{B}_u\mathbf{C}_v = (\mathbf{N}_s\mathbf{C}_s)\mathbf{B}_r - (\mathbf{N}_s\mathbf{B}_s)\mathbf{C}_r.$$

C. Geometric methods [3] and [4].

D. It follows from the definition of the vector product that $\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$ is a vector parallel to the plane of $\mathbf{B}$ and $\mathbf{C}$. Consequently,

$$(2)\qquad \mathbf{N}\times(\mathbf{B}\times\mathbf{C}) = k_1\mathbf{B} + k_2\mathbf{C},$$

where k_1, and k_2 are scalars to be determined. On taking the scalar product of (2) with $\mathbf{N}$, we find that

$$0 = k_1\mathbf{N}\cdot\mathbf{B} + k_2\mathbf{N}\cdot\mathbf{C}. \tag{3}$$

Therefore,

$$\mathbf{N}\times(\mathbf{B}\times\mathbf{C}) = \lambda(\mathbf{N},\mathbf{B},\mathbf{C})[(\mathbf{N}\cdot\mathbf{C})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{C}], \tag{4}$$

where $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is a scalar function of $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, to be determined. It will be assumed here that none of the vectors $\mathbf{N}$, $\mathbf{B}$, or $\mathbf{C}$ is zero, and that $\mathbf{N}$ is not parallel to $\mathbf{B}\times\mathbf{C}$, since in these cases (4) will hold regardless of the value of $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$. The six following proofs all start from this point but differ in the method of proving that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = 1$.

D_1. In this method, [5], $\mathbf{C}$ is resolved into two components $\mathbf{D}$ and $\mathbf{E}$ such that $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{E}$ are not coplanar. Then by (4) it follows that

$$\begin{aligned}\mathbf{N}\times(\mathbf{B}\times[\mathbf{D}+\mathbf{E}]) &= \lambda(\mathbf{N},\mathbf{B},\mathbf{C})[\mathbf{N}\cdot(\mathbf{D}+\mathbf{E})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})(\mathbf{D}+\mathbf{E})]\\ &= \lambda(\mathbf{N},\mathbf{B},\mathbf{C})[(\mathbf{N}\cdot\mathbf{D})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{D}]\\ &\quad + \lambda(\mathbf{N},\mathbf{B},\mathbf{C})[(\mathbf{N}\cdot\mathbf{E})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{E}],\end{aligned} \tag{5}$$

but also

$$\begin{aligned}\mathbf{N}\times(\mathbf{B}\times[\mathbf{D}+\mathbf{E}]) &= \mathbf{N}\times(\mathbf{B}\times\mathbf{D}) + \mathbf{N}\times(\mathbf{B}\times\mathbf{E})\\ &= \lambda(\mathbf{N},\mathbf{B},\mathbf{D})[(\mathbf{N}\cdot\mathbf{D})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{D}]\\ &\quad + \lambda(\mathbf{N},\mathbf{B},\mathbf{E})[(\mathbf{N}\cdot\mathbf{E})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{E}].\end{aligned}$$

Whence it follows that

$$\lambda(\mathbf{N},\mathbf{B},\mathbf{C}) = \lambda(\mathbf{N},\mathbf{B},\mathbf{D}) = \lambda(\mathbf{N},\mathbf{B},\mathbf{E}). \tag{6}$$

The conclusion then improperly drawn in [5] was that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is independent of $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, and thus is a constant. Then by choosing $\mathbf{N}=\mathbf{k}$, $\mathbf{B}=\mathbf{j}$, and $\mathbf{C}=\mathbf{k}$, $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is determined to be equal to 1. The conclusion to be drawn, however, is not that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is independent of $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, but that it is independent of $\mathbf{C}$. (This latter conclusion will still hold even if one of the two vectors $[(\mathbf{N}\cdot\mathbf{D})\mathbf{B}-(\mathbf{N}\cdot\mathbf{B})\mathbf{D}]$, $[(\mathbf{N}\cdot\mathbf{E})\mathbf{B}-(\mathbf{N}\cdot\mathbf{B})\mathbf{E}]$ is zero). To complete the proof, we note that since

$$\lambda(\mathbf{N},\mathbf{B},\mathbf{C}) = \lambda(\mathbf{N},\mathbf{C},\mathbf{B}) \qquad \text{[from (4)]}, \tag{7}$$

it follows that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is also independent of $\mathbf{B}$. By choosing $\mathbf{N}=\mathbf{i}n_1+\mathbf{j}n_2+\mathbf{k}n_3$, $\mathbf{B}=\mathbf{j}$, $\mathbf{C}=\mathbf{k}$, we find that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})=1$.

D_2. Since $\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$ and $[(\mathbf{N}\cdot\mathbf{C})\mathbf{B}-(\mathbf{N}\cdot\mathbf{B})\mathbf{C}]$ are linear distributive functions of each of the vectors $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, it follows that if $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is equal to 1 for every combination of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, it is also equal to 1 for all vectors $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, [6]. This is easily verified.

D_3. In this method, [7], it is first shown that $\lambda(\mathbf{N}, \mathbf{N}, \mathbf{C})=1$. Then using

this result it is shown that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = 1$.

D_4. Here, [8], a special coordinate system is set up such that

$$\mathbf{B} = b_1\mathbf{i}, \qquad \mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j}, \qquad \mathbf{N} = \mathbf{i}n_1 + \mathbf{j}n_2 + \mathbf{k}n_3. \tag{8}$$

It then follows that after substituting (8) in (4) that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = 1$. However, we should add the statement, [9], that if a tensor equation is true in one coordinate system it is true in all coordinate systems.

In methods D_5 and D_6 which follow, the proofs presented are believed to be new.

D_5. Since $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$, it follows that

$$\mathbf{M}\cdot\mathbf{N}\times(\mathbf{B}\times\mathbf{C}) = -\mathbf{C}\cdot\mathbf{B}\times(\mathbf{M}\times\mathbf{N}) = -\mathbf{N}\cdot\mathbf{M}\times(\mathbf{B}\times\mathbf{C}), \tag{9}$$

where $\mathbf{M}$ is chosen such that $\mathbf{M}\times(\mathbf{B}\times\mathbf{C}) \neq 0$. Expanding (9) by means of (4) we get

$$\begin{aligned} &\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})[(\mathbf{N}\cdot\mathbf{C})(\mathbf{M}\cdot\mathbf{B}) - (\mathbf{N}\cdot\mathbf{B})(\mathbf{M}\cdot\mathbf{C})] \\ &\qquad = \lambda(\mathbf{B}, \mathbf{M}, \mathbf{N})[(\mathbf{N}\cdot\mathbf{C})(\mathbf{M}\cdot\mathbf{B}) - (\mathbf{N}\cdot\mathbf{B})(\mathbf{M}\cdot\mathbf{C})] \\ &\qquad = \lambda(\mathbf{M}, \mathbf{B}, \mathbf{C})[(\mathbf{N}\cdot\mathbf{C})(\mathbf{M}\cdot\mathbf{B}) - (\mathbf{N}\cdot\mathbf{B})(\mathbf{M}\cdot\mathbf{C})]. \end{aligned} \tag{10}$$

From this latter equation and (7) it follows that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is constant. By choosing $\mathbf{N} = \mathbf{k}$, $\mathbf{B} = \mathbf{j}$, and $\mathbf{C} = \mathbf{k}$, we find that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = 1$. (For the case $\mathbf{M}\times(\mathbf{B}\times\mathbf{C}) = 0$, (4) with $\mathbf{N}$ replaced by $\mathbf{M}$ will hold for any value of $\lambda(\mathbf{M}, \mathbf{B}, \mathbf{C})$).

D_6. If we let

$$\mathbf{P} = \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}), \tag{11}$$

then $\mathbf{P}\cdot\mathbf{A} = \mathbf{P}\cdot\mathbf{B} = \mathbf{P}\cdot\mathbf{C} = 0$. Thus if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are noncoplanar vectors, $\mathbf{P} = 0$. However, if $\mathbf{A} = m\mathbf{B} + n\mathbf{C}$, then by direct substitution into (11), $\mathbf{P} = 0$. If we expand (11) by means of (4) we find that

$$\lambda(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \lambda(\mathbf{B}, \mathbf{C}, \mathbf{A}) = \lambda(\mathbf{C}, \mathbf{A}, \mathbf{B}), \tag{12}$$

provided that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq 0$. (If $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = 0$, then (4) with $\mathbf{N}$ replaced by $\mathbf{A}$ will hold for any value of $\lambda(\mathbf{A}, \mathbf{B}, \mathbf{C})$). Also, since $\mathbf{N}\cdot\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = -\mathbf{A}\cdot\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$, it follows that $\lambda(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})[\mathbf{N}\cdot\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq 0]$. Thus $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is a constant and thus equal to 1.

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A CONSIDERATION OF ANNUITY FORMULAS

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One of the mathematics courses taught at the freshman level in many colleges is Mathematics of Finance (or Mathematics of Investment). It is highly desirable for the student to have an understanding of the fundamental operations, procedures, and techniques in the course, but it should also be desirable for him to have a set of formulas, which if applied intelligently, would enable him to work problems rapidly. This is especially true in the treatment of annuities certain. It is toward a successful way to present these formulas that this note is directed.* The author has found the following table of formulas to be of very considerable help to the students in his classes. The formulas as presented in the following table have these advantages.

(1) All formulas needed for the treatment of annuities certain in the ordinary freshman course are collected in one place. (The formulas as given here do not make allowances for perfectly general annuities, but these formulas do cover the three primary cases.)

(2) In any one of the six classifications in which both sum and present value formulas are given in the table below, the formulas possess complete similarity. That is, the only difference between formulas is in the $a_{\overline{n}|i}$ or $s_{\overline{n}|i}$ symbol.

(3) In every instance the formulas are the basic formulas for Case I, ordinary annuities (defined later) with additional factors being included because of different data. The particular arrangement of factors is an aid to the student's memory.

(4) In every instance the formulas use the same basic symbols with which the student is already familiar, with the exception of j_p (which is defined below and to which the student may have been introduced in considering nominal rates of interest) and $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ which are defined early in the treatment of Case I annuities. This use of basic symbols eliminates the necessity for the student to memorize a new set of symbols for every new set of data, as is true in some treatments of the subject.

(5) Transition from row to row or column to column of the table is quite simply performed, which is a definite aid to memory.

(6) The formulas for deferred annuities are based on data which enables the student to treat every deferred annuity as an ordinary annuity.

(7) To be sure, other forms of these formulas may lead to simpler computation in certain types of problems, but the formulas as given here lend themselves easily to solution for the periodic payment by a method involving multiplication instead of division. It seems that this more than compensates for the additional work necessitated in certain other types of problems.

* W. L. Hart (vol. 35, p. 358) and C. C. Wylie (vol. 36, p. 327) have both written on this topic in this MONTHLY, but the author has found the system herein presented to be more satisfactory in his classes.

There are textbooks available which make some effort in the direction of this note, but the author has found none which goes as far as giving a complete table of formulas,* and those which have a summary of formulas seem to give the impression that a summary is needed due to the very large number of different symbols used.

For the formulas in the table the following definitions are used:

(1) Case I annuity—an annuity in which payment periods and interest conversion periods coincide;

(2) Case II annuity—an annuity in which there are k interest conversion periods (k an integer greater than 1—this defines k) in one payment period;

(3) Case III annuity—an annuity in which there are p payment periods (p an integer greater than 1—this defines p) in one interest conversion period.

As a final word before giving the table of formulas, it may be noted that an added advantage is that the quantities involved in these formulas are explicitly stated and may be found in any standard set of tables included in most textbooks which treat this subject.

Table of Annuity Formulas

	Ordinary Annuity	Annuity Due	Deferred Annuity
Case I	$A=Ra_{\overline{n}\mid i}$ $S=Rs_{\overline{n}\mid i}$	$A=Ra_{\overline{n}\mid i}(1+i)$ $S=Rs_{\overline{n}\mid i}(1+i)$	$A=Ra_{\overline{n}\mid i}(1+i)^{-m}$
Case II	$A=Ra_{\overline{n}\mid i}\cdot\dfrac{1}{s_{\overline{k}\mid i}}$ $S=Rs_{\overline{n}\mid i}\cdot\dfrac{1}{s_{\overline{k}\mid i}}$	$A=Ra_{\overline{n}\mid i}\cdot\dfrac{1}{s_{\overline{k}\mid i}}(1+i)^{k}$ $S=Rs_{\overline{n}\mid i}\cdot\dfrac{1}{s_{\overline{k}\mid i}}(1+i)^{k}$	$A=Ra_{\overline{n}\mid i}\cdot\dfrac{1}{s_{\overline{k}\mid i}}(1+i)^{-m}$
Case III	$A=Ra_{\overline{n}\mid i}\cdot p\cdot\dfrac{i}{j_p}$ $S=Rs_{\overline{n}\mid i}\cdot p\cdot\dfrac{i}{j_p}$	$A=Ra_{\overline{n}\mid i}\cdot p\cdot\dfrac{i}{j_p}(1+i)^{1/p}$ $S=Rs_{\overline{n}\mid i}\cdot p\cdot\dfrac{i}{j_p}(1+i)^{1/p}$	$A=Ra_{\overline{n}\mid i}\cdot p\cdot\dfrac{i}{j_p}(1+i)^{-m}$

* A table similar in a few respects to the one given here is to be found in Simpson, Pirenian, and Crenshaw, Mathematics of Finance (3rd ed.), Prentice-Hall, Inc., 1951, p. 225. For convenience, the notation used in this paper corresponds to that used in the foregoing book, pp. 224, 225, with the following elaborations: R = the periodic payment (not rent); n = the number of interest conversions in the term of the annuity; p, as defined in Case III; and $j_p=p[(1+i)^{1/p}-1]$.

THE PROBABILITY INTEGRAL

A. J. COLEMAN, University of Toronto

The author was recently faced with the problem of evaluating the probability integral for a class which had not studied double integration, so the well-known method reproduced on p. 129 of Feller's *Probability Theory and its Applications* was not suitable. The following is the author's solution of his problem. He would be glad to learn of more elementary ones.

Consider the integral,

$$I_m = \int_0^\infty x^m e^{-nx^2/2} dx$$

where m and n are integers. Integrating by parts, we see that

$$I_m = \frac{m-1}{n} I_{m-2}, \qquad (m > 1, n > 0). \tag{1}$$

For any real t,

$$\int_0^\infty x^m (x-t)^2 e^{-nx^2/2} dx > 0.$$

Therefore, $I_{m+2} - 2tI_{m+1} + t^2 I_m > 0$ for all t. Hence,

$$I_{m+1} < \sqrt{I_m I_{m+2}}.$$

If we set m equal to n and $n-1$, and use (1) for m equal to $n+1$ and $n+2$, we obtain the inequalities

$$I_{n+2} > I_{n+1} > I_n. \tag{2}$$

For $n = 2k+1$, where k is any positive integer, the recursion formula (1) easily gives

$$I_{2k+1} = \frac{2 \cdot 4 \cdots (2k)}{(2k+1)^{k+1}}$$

$$I_{2k+2} = \frac{1 \cdot 3 \cdot 5 \cdots (2k+1)}{(2k+1)^{k+1}} I_0$$

$$I_{2k+3} = \frac{2 \cdot 4 \cdots (2k)(2k+2)}{(2k+1)^{k+2}}.$$

A change of variables shows that

$$I = \int_0^\infty e^{-u^2/2} du = \sqrt{2k+1}\, I_0.$$

Substituting these expressions in (2) gives,

$$\left(1+\frac{1}{2k+1}\right)\frac{2\cdot 4\cdot\cdots\cdot(2k)}{1\cdot 3\cdot\cdots\cdot(2k-1)}\,\frac{1}{\sqrt{2k+1}} > I > \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{1\cdot 3\cdot\cdots\cdot(2k-1)}\,\frac{1}{\sqrt{2k+1}}.$$

But Wallis' formula* states that the final expression above approaches $\sqrt{\pi/2}$ as k approaches infinity. The expression on the left approaches the same limit. Hence,

$$\int_0^\infty e^{-u^2/2}du = \sqrt{\frac{\pi}{2}}.$$

* See Courant and Robbins, What is Mathematics? pp. 509–510, for details.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1141. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Find the radius of the circle inscribed in the mixtilinear triangle formed by the two legs of a given right triangle ABC and the semi-circumference described externally upon the hypothenuse AB.

E 1142. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find the semi-vertical angle of a right circular cone if three generating lines make angles of 2α, 2β, 2γ, with each other.

E 1143. *Proposed by C. D. Olds, San Jose State College, Calif.*

If the roots of the equation

$$a_0x^n - na_1x^{n-1} + \tfrac{1}{2}n(n-1)a_2x^{n-2} - \cdots + (-1)^n a_n = 0$$

are positive and distinct, prove that $a_r a_{n-r} > a_0 a_n$, $r = 1, 2, \cdots, n-1$.

E 1144. *Proposed by A. S. Hendler, Rensselaer Polytechnic Institute, N. Y.*

For what positive values of a is $\log_a b < b$ for all positive b?

E 1145. *Proposed by B. A. Hausmann, West Baden College, Indiana*

Let $I=(m+n-1)!/m!n!$, where m and n are natural numbers. Find a necessary and sufficient condition for I to be an integer.

SOLUTIONS

A "Faded Document" Division

E 1111 [1954, 258]. *Proposed by P. L. Chessin, Westinghouse Electric Corporation*

Our good friend and eminent numerologist, Professor Euclide Paracelso Bombasto Umbugio, has been busily engaged testing on his desk calculator the $81\cdot 10^9$ possible solutions to the problem of reconstructing the following exact long division in which the digits indiscriminately were each replaced by x save in the quotient where they were almost entirely omitted.

$$
\begin{array}{rl}
 & 8 \\
\hline
x\,x\,x\,) & x\,x\,x\,x\,x\,x\,x\,x \\
 & \underline{x\,x\,x} \\
 & x\,x\,x\,x \\
 & \underline{x\,x\,x} \\
 & x\,x\,x\,x \\
 & \underline{x\,x\,x\,x}
\end{array}
$$

Deflate the Professor! That is, reduce the possibilities to $(81\cdot 10^9)^9$.

Solution by W. B. Carver, Cornell University. If we denote the divisor by d, we have $8d<1000$, $d<125$. Then, since $7d<900$, it follows that the first digit in the quotient is 8, and that the quotient is 80809. Then, since $(80809)d$ must be greater than 10,000,000, we have $d>123$. Hence $d=124$.

Also solved by P. M. Berry, D. G. Brennan, R. L. Caskey, Monte Dernham, William Douglas, S. H. Eisman, P. C. Fife, Jean Gamzon, W. V. Gamzon, H. W. Gould, W. T. Grant, R. E. Greenwood, T. N. E. Greville, Fred Gruenberger, R. K. Guy, J. D. Haggard, J. W. Hamblen, H. J. Hauer, Joy Heller, Boyd Henry, H. K. Hilton, R. T. Hood, R. E. Horton, A. R. Hyde, Virginia Johnson, Edgar Karst, J. M. Kingston, H. R. Leifer, Octave Levenspiel, D. C. B. Marsh, J. H. Means, B. E. Mitchell, George Mott, T. F. Mulcrone, John Murtha, E. J. Musch, David Muskat, J. B. Muskat, C. S. Ogilvy, H. J. Osner, J. V. Pennington, Walter Penney, P. A. Piza, Alan Plait, N. Popiel, Jr., L. A. Ringenberg, Max Rosenberg, Azriel Rosenfeld, David Rothman, E. P. Rozycki, W. M. Sanders, William Saul, R. E. Shafer, S. W. Shelton, N. Shklov, Malcolm Smith, R. A. Spong, R. B. Stein, R. P. Tapscott, A. J. Tingley, C. W. Trigg, George Walker, Robert Wallach, E. G. Walsh, W. D. Ward, R. H. Wilson, Jr., Monica Wyzalek, and the proposer. Late solution by E. S. Johnson.

Several solvers located the problem and the answer, with no analysis or proof of uniqueness, as Problem 33 in Joseph de Grazia, *Math is Fun*, Gresham Press, New York (1948), pp. 34, 35, 119. Apparently the problem is considerably older, however, for a number of solvers recalled it from 15 to 20 years back.

J. B. Muskat remarked that if the middle 8 in the quotient had not been given, there would have been 163 possible solutions. Plait remarked that if the given 8 in the quotient had been the last digit of the quotient, instead of the middle one, there would have been exactly 10 different solutions.

The proposer stated that if Professor Umbugio were to perform one long division calculation a second and work an eight hour day every day of the week, without summer vacations, his technique would take about 77 years!

Trigg observed that E 1111 contains as many ones as the Professor has names, 4, and this times the number of letters in the Professor's complete name, 31, gives the divisor of the problem. Thus the number of the Professor's names and the number of letters in these names are congruent modulo 9. Then the sum, 25, of the digits of the quotient, the last two digits of the dividend, the modulus, the number of the Professor's names, and the Professor are consecutive squares. See E 961 [1951, 700].

The Tac-locus of Two Parabolic Pencils of Circles

E 1112 [1954, 259]. *Proposed by L. C. Graue, Sacramento State College*

Consider two families of circles, one tangent at the origin to the x-axis and the other tangent at the point (1, 1) to a line of slope m. Find the locus of the points of tangency of the two families.

I. *Solution by Hüseyin Demir, Zonguldak, Turkey.* Let (C) be the circle tangent to Ox at O, d the line containing $A(1, 1)$, (U_1) and (U_2) the circles tangent to d at A and to (C) at T_1 and T_2 respectively. Invert the figure with respect to O as center. Under the inversion (C), d, A, (U_1), (U_2), T_1, T_2 invert into c, (D), A', (U_1'), (U_2'), T_1', T_2' respectively. Evidently circle (D) passes through O, and line c is parallel to Ox. Also, (U_1') and (U_2') are tangent to (D) at A', and to c at T_1' and T_2'. Since the tangent c to (U_1'), $U_2')$ at T_1', T_2' keeps a fixed direction, the loci of T_1', T_2' are straight lines $A'T_1'$, $A'T_2'$. Hence the required loci of T_1 and T_2 are two circles. Each of these circles passes through A and the center of inversion O.

We now show that these two circles are orthogonal by showing that $A'T_1'$, $A'T_2'$ are perpendicular to each other. Consider the common tangent to the circles (D), (U_1'), (U_2'); it bisects $T_1'T_2'$ at I. Since $IA' = IT_1' = IT_2'$, triangle $T_1'A'T_2'$ is a right triangle and $A'T_1'$, $A'T_2'$ are perpendicular.

The result may be stated as a theorem: *The locus of the points of tangency of the circles of any two parabolic pencils of circles consists of two orthogonal circles each passing through the base points of the two pencils.*

II. *Solution by Chih-yi Wang, University of Minnesota.* The general equations of the two families are, respectively,

$$x^2 + y^2 - 2ay = 0, \tag{1}$$

$$(x - m - 1 + mk)^2 + (y - k)^2 = (1 + m^2)(1 - k)^2, \tag{2}$$

where a and k are parameters. Since the points of tangency must lie on the two circles and on the lines joining the centers, we must also have

$$(k - a)x - (m + 1 - mk)(y - a) = 0. \tag{3}$$

Eliminating a and k from (1), (2), (3) we obtain, after simplification,

$$m[(x^2 + y^2)^2 - 2(x + y)(x^2 + y^2) + 4xy] + 2(x - y)(x^2 + y^2 - x - y) = 0. \tag{4}$$

Case I ($m \neq 0$). We obtain the equations of two orthogonal circles:

$$x^2 + y^2 - (1 - 1/m \pm \sqrt{1 + m^2}/m)x - (1 + 1/m \mp \sqrt{1 + m^2}/m)y = 0. \tag{5}$$

If the line passing through (1, 1) is parallel to the y-axis, we let $m \to \infty$ in (5).

Case II ($m = 0$). We obtain from (4) equations of a circle and one of its diametral lines:

$$x^2 + y^2 - x - y = 0 \quad \text{and} \quad x - y = 0.$$

The same results hold if we eliminate a and k from (1) and (2) by differentiating and equating dy/dx, without using (3).

Also solved by B. H. Bissinger, W. B. Carver, Michael Goldberg, M. S. Klamkin, Beckham Martin, C. S. Ogilvy, O. J. Ramler, R. E. Shafer, N. Shklov, and the proposer.

Bounds on a Solution of a Special Riccati Equation

E 1113 [1954, 259]. *Proposed by Peter Treuenfels, Ballistic Research Laboratories, Aberdeen Proving Ground*

Let $y(x)$ denote that solution of the differential equation $dy/dx = x^2 + y^2$ which passes through the origin. Show that $y(1) < 23/53$.

Solution by R. E. Shafer, University of California. We can evaluate upper and lower bounds of $y(x)$ by successive approximation. Clearly $dy/dx \geqq 0$ and, for a fixed value of x, dy/dx is an increasing function of y. Let

$$dy_n(x)/dx = y_{n-1}^2(x) + x^3. \tag{1}$$

For generating a set of lower bounds $y_0 < y_1 < y_2 < \cdots$ for $0 \leqq x \leqq 1$ let $y_0(x) = 0$. Using equation (1) and imposing the condition $y(0) = 0$, we obtain

$$y_1 = x^4/3, \qquad y_2 = x^4/3 + x^7/63.$$

Hence we may use $y_2(1) = 22/63$ as a lower bound to $y(1)$.

For generating a set of upper bounds $y_0 > y_1 > y_2 > \cdots$ for $0 \leqq x \leqq 1$ let

$y_0(x)=x$. Using equation (1) and imposing the condition $y(0)=0$, we obtain

$$y_1 = 2x^3/3, \qquad y_2 = x^3/3 + 4x^7/63,$$

$$y_3 = x^3/3 + x^7/63 + 8x^{11}/11\cdot 189 + 16x^{15}/15\cdot 63^2.$$

Hence we may use $y_3(1)<23/63$ as an upper bound to $y(1)$.

It follows that $22/63<y(1)<23/63$.

Also solved by A. N. Aheart, B. H. Bissinger, Christian Blatter, C. N. Campopiano, W. B. Carver, J. E. Darraugh, William Dite and Malcolm Smith (jointly), I. A. Dodes, P. C. Fife, E. K. Greatrix, Louisa Grinstein, P. G. Hodge, Jr., A. R. Hyde, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, T. F. Mulcrone, C. S. Ogilvy, F. D. Parker, L. A. Ringenberg, David Rothman, D. C. Russell, W. M. Sanders, O. E. Stanaitis, Chih-yi Wang, J. E. Wilkins, Jr., the proposer, and someone with an illegible signature.

Marsh and Stanaitis showed that $y(1)<4/11$, which is a closer upper bound than 23/63. Actually $y(1)=0.350168^+$. In view of the fact that 23/63 is an upper bound it might be thought that the 23/53 of the proposed problem was a misprint; this was not the case.

Pie and Plate Problem

E 1114 [1954, 259]. *Proposed by Viktors Linis, University of Saskatchewan*

What is the diameter of the smallest circular plate on which a semicircular pie can be placed if the pie is allowed to be cut in sectors of the same radius as the pie?

Solution by W. B. Carver, Cornell University. Let the radius of the semicircular pie be r, its area $\pi r^2/2$. Cut the pie into a very large number of equal sectors. By laying a number of sectors adjacent to one another, with radial sides coincident but oppositely directed, and by removing half of one end-sector to the other end of the array, an approximate rectangle can be formed. Let us fit the sectors of the pie into three such rectangles, one rectangle r by $r+2x$ and two others r by x. These rectangles can now be fitted together to form an approximate Greek cross (gotten by cutting squares of side x from each corner of a larger square of side $r+2x$), which can be placed on a circular plate whose diameter is equal to a diagonal of the big rectangle. This diameter is easily calculated to be

$$d = (r/4)\sqrt{\pi^2 + 4\pi + 20} = 0.6286r, \text{ approximately.}$$

It is clear that by using a sufficiently large number of sectors we can, in this fashion, place the pie on a plate of diameter

$$(r/4)\sqrt{\pi^2 + 4\pi + 20} + \epsilon,$$

where ϵ is an arbitrarily small positive number.

It is of course possible that some entirely different method of cutting and arranging the pieces of the pie would require a plate of diameter less than the limiting diameter of this method.

Also solved by R. K. Guy, A. R. Hyde, C. S. Ogilvy, and the proposer.

The proposer expressed interest in the best solution for a given (small) number n of pieces (equal or not?). He found results for a few cases:

$n = 1, 2, 3$	$d = 2r$
$n = 4$	$d = \sqrt{3}r = 1.7320r$
$n = 5$	$d = 1.7016r$ (the best?)
$n = 6$	$d = 1.7390r$
$n = 7$	$d = 1.8080r$
$n = 8$	$d = 1.7320r$ (as for $n = 4$).

Carver, by arranging 90 equal sectors into a Greek cross, as in his solution above, found for this case $d = 1.6336r$.

A Poristic Property

E1115 [1954, 259]. *Proposed by R. M. Gordon, China Lake, California*

(1) Let $Q_1Q_2Q_3Q_4$ be a (not necessarily convex) plane quadrilateral. On its sides construct similar isosceles triangles $Q_vP_vQ_{v+1}$ (with $Q_5 \equiv Q_1$), having arbitrary base angles θ. The angles $Q_{v+1}Q_vP_v (=\theta)$ are oriented alternately clockwise and counterclockwise from the adjacent sides, $Q_{v+1}Q_v$, of the quadrilateral. Show that P_1, P_2, P_3, P_4, the vertices of the isosceles triangles, are the vertices of a parallelogram.

(2) Let $P_1P_2P_3P_4$ be a plane quadrilateral. On its vertices construct similar isosceles triangles $Q_vP_vQ_{v+1}$ (with $Q_5 \equiv Q_1$) having vertex angles $Q_vP_vQ_{v+1}$ and base angles θ. The angles $Q_{v+1}Q_vP_v$ $(=\theta)$ are oriented alternately clockwise and counterclockwise in adjacent triangles. Show that, for arbitrary θ, the bases of the isosceles triangles are the sides of infinitely many quadrilaterals $Q_1Q_2Q_3Q_4$ provided that $P_1P_2P_3P_4$ is a parallelogram, and that if $P_1P_2P_3P_4$ is not a parallelogram then there exists a unique quadrilateral $Q_1Q_2Q_3Q_4$.

Solution by W. B. Carver, Cornell University. The final statement in paragraph (2) of the problem should read, "if $P_1P_2P_3P_4$ is not a parallelogram then there exists *no* quadrilateral $Q_1Q_2Q_3Q_4$."

(1) Let the points Q_i and P_i of the problem correspond respectively to the complex numbers q_i and p_i in the usual representation of complex numbers in the plane; and let $t = e^{2i\theta}$, where θ is the base angle of the problem. Then since $Q_1P_1Q_2$ is an isosceles triangle with base angles θ, we must have

$$t(p_1 - q_1) + (p_1 - q_2) = 0,$$
$$(t + 1)p_1 = q_2 + q_1 t,$$

for the clockwise orientation; and

$$t(p_1 - q_2) + (p_1 - q_1) = 0,$$
$$(t + 1)p_1 = q_1 + q_2 t,$$

for the counterclockwise orientation. Thus the p_i are determined by the set of equations

$$\text{(A)} \qquad \begin{aligned} (t + 1)p_1 &= q_2 + q_1 t, \\ (t + 1)p_2 &= q_2 + q_3 t, \\ (t + 1)p_3 &= q_4 + q_3 t, \\ (t + 1)p_4 &= q_4 + q_1 t. \end{aligned}$$

Multiplying the second and fourth equations by -1 and adding, we have

$$(t + 1)(p_1 - p_2 + p_3 - p_4) = 0.$$

Since $t+1=0$ is impossible (this would make the base angle θ a right angle), it follows that

$$\text{(B)} \qquad p_1 - p_2 + p_3 - p_4 = 0,$$

and thus $P_1P_2P_3P_4$ must be a parallelogram.

(2) Now if the p_i are given, we have the set of equations (A) to determine the q_i. But this set of equations is not satisfied by any set of values of the q_i unless condition (B) is satisfied. Hence if $P_1P_2P_3P_4$ is not a parallelogram there can be *no* quadrilateral $Q_1Q_2Q_3Q_4$ satisfying the requirements of the problem. If condition (B) is satisfied, we may take one of the q_i arbitrarily and the remaining q_i are then uniquely determined, giving infinitely many quadrilaterals $Q_1Q_2Q_3Q_4$ as stated in the problem.

Also solved by Hüseyin Demir (as above) and C. S. Ogilvy (by elementary geometry).

The two parts of this problem are generalizations of the familiar theorems: (1) The midpoints of the sides of a plane (not necessarily convex) quadrilateral are the vertices of a parallelogram. (2) The vertices of a plane quadrilateral P are the midpoints of the sides of infinitely many quadrilaterals Q provided that P is a parallelogram; if P is not a parallelogram there exists no quadrilateral Q.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4613. *Proposed by Joseph Lehner and G. Milton Wing, Los Alamos Scientific Laboratory*

Let $\phi(t)$ be integrable over (0, 1) and suppose, for $z=re^{i\theta}$, that

$$F(z) = \int_0^1 e^{\alpha z t}\phi(t)dt, \qquad R\alpha > 0$$

satisfies

$$\int_0^\infty |\, F(r)\,|^2 dr < \infty.$$

Show that $\phi = 0$ almost everywhere.

4614. *Proposed by H. F. Sandham, Institute for Advanced Studies, Ireland*

Prove that

$$\int_0^1 (1-x)(1-x^2)(1-x^3)\cdots dx = \pi\sqrt{\frac{48}{23}}\,\frac{\sinh \tfrac{1}{2}\pi\sqrt{23}}{\cosh \tfrac{1}{2}\pi\sqrt{23}}.$$

4615. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Define

$$F(x) = \cos x \cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4}\cdots.$$

Prove that $F(x)\to 0$ and $x\to+\infty$. Prove in fact that $F(x)$ falls off exponentially.

4616. *Proposed by A. C. Cohen, Jr., University of Georgia*

Show that the compound normal distribution function

$$f(x) = \frac{1}{2\sigma\sqrt{2\pi}}\left[\exp\left\{-\tfrac{1}{2}\left(\frac{x-m_1}{\sigma}\right)^2\right\} + \exp\left\{-\tfrac{1}{2}\left(\frac{x-m_2}{\sigma}\right)^2\right\}\right]$$

is bimodal if $|m_1-m_2| > 2\sigma$ and unimodal otherwise.

4617. *Proposed by Albert Wilansky, Lehigh University*

Let $\{u_n\}$, $\{v_n\}$ be real sequences, $n=2, 3, 4, \cdots$, such that

$$\sum (u_n^2 + v_n^2) < 1.$$

Let f, g be real continuous functions of two real variables x, y, $x^2+y \leqq 1$; such that

$$f(\cos\theta, \sin\theta) = \cos\theta - \sum (u_n \cos n\theta + v_n \sin n\theta)$$
$$g(\cos\theta, \sin\theta) = \sin\theta - \sum (v_n \cos n\theta - u_n \sin n\theta).$$

Show that f, g vanish simultaneously at some point (x, y) for which $x^2+y^2<1$.

SOLUTIONS

Derangement of a Familiar Series

4552 [1953, 482]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

What derangement of terms of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

will produce a sum which is rational?

Solution by V. C. Harris, San Diego State College, California. If neither the order in which the positive terms occur nor the order in which the negative terms occur is changed, but if the terms are rearranged so that k is the limit of the ratio of the number of positive terms to the number of negative terms in the first n terms, then the alteration L in the sum of the series is $L=\frac{1}{2}\log k$. (This result is due to Pringsheim; see Bromwich, *An Introduction to the Theory of Infinite Series*, Second Edition, Revised, p. 76.) Since the sum of the given series is $\log 2$, the sum after alteration is $\log 2+\frac{1}{2}\log k$. Setting $k=K^2/4$, the sum becomes $\log K$. If this is rational, say m/n, then $k=K^2/4=(1/4)\exp(2m/n)$. To effect the derangement, set up any sequence of rationals which has limit $(1/4)\exp(2m/n)$, *e.g.* convergents of its continued fraction expansion, and take a number of positive terms equal to the numerator followed by a number of negative terms equal to the denominator, of the successive terms of the sequence. In particular, for $k=1/4$, we have

$$0 = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \cdots.$$

Also solved by L. C. Graue, Erich Michalup, C. D. Olds, S. Parameswaran, L. A. Ringenberg, and the Proposer.

Inequality

4553 [1953, 554]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Consider any sequence $\{a_n\}$ of real numbers. Prove

$$\sum_{n=1}^{\infty} a_n \leqq \sqrt{2}\sum_{n=1}^{\infty}\sqrt{\frac{a_n^2 + a_{n+1}^2 + \cdots}{n}}.$$

Solution by Chih-yi Wang, University of Minnesota. In Hardy, Littlewood and Pólya, *Inequalities* (2nd ed.), p. 255, theorem 345 states that if $0<p<1$, then

$$\sum_{n=1}^{\infty}\left(\frac{\alpha_n + \alpha_{n+1} + \cdots}{n}\right)^p > p^p\sum_{n=1}^{\infty}\alpha_n^p.$$

The given problem is a special case with $\alpha_n = a_n^2$ for all n and $p=\frac{1}{2}$. Equality holds if $a_n=0$ for all n.

Also solved by O. E. Stanaitis and the Proposer.

Solution of a Differential Equation

4554 [1953, 554]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, New York.*

Find the general solution of the differential equation

$$\left(\frac{dy}{dx} + Py\right)^P = R\left(\frac{dy}{dx} + Qy\right)^Q$$

where P, Q and R are constants.

Solution by Omar Ali Siddiqi, Aligarh University, India. Taking logarithms and differentiating and simplifying, we have, if $P \neq Q$,

$$y'\{y'' + (P+Q)y' + PQy\} = 0,$$

giving the general solution $y = Ae^{-Px} + Be^{-Qx}$. The constants A, B are not independent but, on substituting in the differential equation,

$$(P-Q)^P B^P = R(Q-P)^Q A^Q.$$

If $P=Q$, and for certain other special cases in which the above does not apply, the solution is immediate.

With little additional difficulty the equation

$$(y' + P_1y)^P = R(y' + Q_1y)^Q$$

can be solved, where no condition on P, Q, P_1, Q_1 is specified. Logarithmic differentiation gives

$$a_1y''y' + a_2y''y + a_3y'^2 + a_4y'y = 0,$$

the a_i being constants expressible in terms of P, Q, P_1, Q_1. If now the dependent variable is changed to $v=y'/y$, the equation becomes

$$(a_1v + a_2)(v' + v^2) + a_3v^2 + a_4v = 0,$$

so that

$$x + A = -\int \frac{(a_1v + a_2)dv}{a_1v^3 + (a_2 + a_3)v^2 + a_4v}.$$

By performing the integration in finite terms or in series and determining the second arbitrary constant, the solution can be completed.

Also solved by W. J. Blundon, L. F. Boron, Emil Grosswald, Karl Guderley and Henry Fettis, R. T. Herbst, M. S. Klamkin, M. Morduchow, Celestine O'Callaghan, M. J. Pascual, L. B. Robinson, R. I. Scibor-Marchocki, O. E. Stanaitis, B. Stanović, F. Underwood, Chih-yi Wang, and the Proposer.

A Ballot Problem Related to Lattice Paths

4556 [1953, 555]. *Proposed by H. D. Grossman, New York City*

Suppose an election results in km votes for A and kn for B. In how many orders may votes be cast so that A's vote is always at least m/n times B's? Prove the following formula

$$p_k = \sum \frac{F_1^{k_1}F_2^{k_2} \cdots}{k_1!k_2! \cdots}$$

in which $k_1+2k_2+ \cdots =k$ and $F_j=\binom{jm+jn}{jm}/j(m+n)$ and the summation is over all partitions of k.

Editorial Note. In a paper with the title, New formula for the number of minimal lattice paths, *The Journal of the Institute of Actuaries*, London, vol. 80, pp. 55–62, M. T. L. Bizley considers the problem in the form: Determine the number of lattice paths from (0, 0) to (km, kn) which do not rise above the line $my=nx$. (A lattice path is a path joining (0, 0) to (x, y) say, composed of horizontal and vertical steps of unit length, the total number of steps being $x+y$.) He provides a proof of Grossman's formula and in addition gives (with proof) the number of paths which do not meet $my=nx$ at all between the end points, and also the number which meet it at a specified number of points.

Summation

4557 [1953, 555]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Let $f(n)$ be the number of ways the integer n can be decomposed into dissimilar factors [*e.g.*, $f(1)=1, f(45)=3$ because $1\cdot 45=3\cdot 15=5\cdot 9$ but not $3\cdot 3\cdot 5$]. Evaluate

$$\sum_{n=0}^{\infty} \frac{f(2n+1)}{(2n+1)^2}.$$

Solution by W. J. Blundon, Memorial University of Newfoundland. For every factorization $k_1k_2k_3\cdots$ of $2n+1$ such that $k_1<k_2<\cdots$, there is associated the term a_n, where $a_n=(2n+1)^{-2}$. Thus the given expression is equal to

$$1+\sum a_1+\sum a_1a_2+\sum a_1a_2a_3+\cdots=\prod_{n=1}^{\infty}(1+a_n)=\prod_{n=1}^{\infty}[1+(2n+1)^{-2}]$$

$$=\frac{1}{2}\prod_{n=0}^{\infty}[1+(2n+1)^{-2}]=\frac{1}{2}\cosh\frac{1}{2}$$

from the well-known infinite product.

Also solved by W. E. Briggs, N. J. Fine, and the Proposer.

An Extension of Pascal's Theorem

4558 [1953, 631]. *Proposed by V. F. Ivanoff, San Francisco*

If an octagon $l_1l_2\cdots l_8$ is inscribed in a conic, then the eight points of intersection of the sides l_i and l_j ($j\equiv i+3$, mod 8) lie on another conic.

I. *Solution by Harley Flanders, University of California, Berkeley.* We denote the vertices of the simple octagon by

$$P_{12}=l_1\cap l_2,\ P_{23}=l_2\cap l_3,\ \cdots,\ P_{81}=l_8\cap l_1,$$

and we set

$$Q_{14}=l_1\cap l_4,\ Q_{25}=l_2\cap l_5,\ \cdots,\ Q_{83}=l_8\cap l_3.$$

The problem is to prove that the Q's are coconical. Evidently it suffices to prove that enough sets of six of these points are on a conic.

Consider the simple hexagon $l_1l_2l_3l_4l_5l_6'$, where $l_6'=P_{56}\cup P_{81}$. By Pascal's theorem $Q'=l_6'\cap l_3$, Q_{14} and Q_{25} are collinear. Next, consider the simple hexagon $Q_{14}Q_{36}Q_{58}Q_{83}Q_{25}Q_{61}$. We have

$$(Q_{14}\cup Q_{25})\cap(Q_{36}\cup Q_{83})=(Q_{14}\cup Q_{25})\cap l_3=Q',$$

$$(Q_{14}\cup Q_{61})\cap(Q_{83}\cup Q_{58})=l_1\cap l_8=P_{81},$$

$$(Q_{36}\cup Q_{61})\cap(Q_{25}\cup Q_{58})=l_6\cap l_5=P_{56},$$

all points on the line l_6'. Hence, by Pascal's theorem again, the vertices of the hexagon are on a conic.

By symmetry, the points Q_{47}, Q_{61}, Q_{83}, Q_{36}, Q_{58}, Q_{14} are also coconical. Combined with the last result, this gives us seven of the Q's on a conic. One more such manipulation and we have all eight of the Q's on this conic.

II. *Solution by H. S. M. Coxeter, University of Toronto.* This is a nice exercise on the principle of coresiduation (H. F. Baker, *Principles of Geometry*, Vol. V, Cambridge, 1933, p. 67). The sixteen common points of the two degenerate quartic curves,

$$l_1l_3l_5l_7 = 0, \qquad l_2l_4l_6l_8 = 0,$$

are "associated" in the sense that every quartic through thirteen of them goes through the remaining three also. One such quartic is

$$ss' = 0,$$

where $s=0$ is the given conic and $s'=0$ is the conic determined by five of the eight intersections described. The remaining three, not lying on $s=0$, must lie on $s'=0$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Theory of equations. By C. C. MacDuffee. New York, John Wiley and Sons, Inc., 1954. 7+120 pages. $3.75.

As the author mentions in his preface, this book is the outgrowth of a one semester course given to junior and senior students. In a little over a hundred pages the author very competently covers the standard topics in the theory of equations. A small amount of abstract algebra is incorporated in the development of the subject.

Chapter 1. Linear systems. The feature of this chapter is the treatment of linear systems without the use of determinants. Indeed determinants are not included in the book. Chapter 2. Rational solutions. After a short introduction to number-theory, the familiar theorems on rational roots of polynomials is developed. Chapter 3. Polynomials. The chapter opens with the abstract definition of (commutative) ring and field. It includes a section on partial fractions. Chapter 4. Real roots. This chapter includes the Budan-Fourier theorem, Descartes' rule of signs, the Sturm theorem, the rule of false position, Newton's method and Horner's method. Chapter 5. Complex roots. The "fundamental theorem of algebra" is proved for polynomials of degree $\leqq 5$. Chapter 6. Relations among the roots. This chapter contains a brief discussion of symmetric polynomials, including the fundamental theorem, Newton's identities and the definition of the discriminant, and the solution of the general cubic and quartic.

Chapter 7. Systems of higher degree. Beginning with Euclidean rings, the chapter includes the following section headings: Eisenstein's criterion, content and primitive part, greatest common divisor, unique factorization, systems of symmetric equations, a general method (for systems), the resultant, the discriminant. Presumably this last chapter will seem a bit difficult to the immature reader.

This is a pleasant and readable little book. The reviewer regrets that the author has not included more material even though there is ample material in the book for a one-semester course. Also a set of interesting miscellaneous problems would have helped round out the book.

L. CARLITZ
Duke University

Introduction to College Mathematics. By C. V. Newsom and Howard Eves. New York, Prentice-Hall, Inc. 1954. 8+408 pages. $5.75.

This book is a second edition of an earlier edition published in 1946 with Newsom as the sole author. It was written with a view to answering the needs of the ever increasing number of college students who are destined to take but one course in mathematics and for whom many, including this reviewer, believe the traditional freshman courses are not suitable.

The material covered is indicated by the following chapter headings: 1. Number and the Operations of Arithmetic. 2. The Arithmetic of Numbers in the Exponential Form. 3. The Arithmetic of Measurement. 4. Logarithms. 5. Progressions of Numbers. 6. Statistics. 7. Combinations and Probability. 8. Functional Relationships. 9. Variation. 10. The Circular Functions. 11. Trigonometry. 12. The Equation. 13. Some Common Curves. 14. A Glimpse of the Calculus.

One of the outstanding features of the book is the abundance of varied and interesting exercises which often are as enlightening as the text itself, as for example, the highly informative exercise on Fibonacci progressions on page 94.

Of especial interest, both as to novelty of treatment and pedagogical significance, is the introduction in Chapter 10 of circular functions as functions of a real variable (directed arc length) rather than as functions of angles. The derivations of the formulas for sine and cosine of sums and differences of arcs are based on an interesting method of analysis attributed to Cauchy. A brief descriptive discussion of harmonic analysis is also given in Chapter 10. Applications of the circular functions to angles are treated in Chapter 11.

An excellent footnote on equivalent equations occurs on page 301 in the section on the parabola. The reviewer is inclined to think that the value to the students of this and several other illuminating footnotes would be materially increased if they were made a part of the text.

The clear treatment of the concept of instantaneous speed and instantaneous rate of change on page 339 is noteworthy.

It appears pedagogically sounder, as well as more in keeping with the logical development, to explain the definition of the multiplication of two negative numbers on the basis of the distributive law, as suggested in example 5 on page 13, rather than on the basis of division by equals.

In the interest of clarity, it would seem advisable to state what geometrical property of a straight line is being taken as a definition in the analytical treatment of the straight line in Chapter 13.

It appears to the reviewer that at least a partial explanation should be given of the basis of the procedure for finding the line of best fit. Such an explanation, for example, as that given on pages 120–122 of Richardson's *Introduction to Statistical Analysis* would seem entirely within the scope of the book and would add to its usefulness.

Everything considered, the authors are to be congratulated on writing a book so admirably adapted to a terminal course in college mathematics.

A. H. DIAMOND
Stevens Institute of Technology

Elements of Statistics. By H. C. Fryer. New York, John Wiley and Sons, Inc., 1954. vii+262 pages.

The author states that the purpose of this book is to provide the college and university student with an introductory course in statistics and that college algebra is the only prerequisite.

After an introduction, methods of summarizing data are studied. A discussion of elementary probability then prepares the student for the study of the binomial and normal distributions and sampling from these distributions. Certain aspects of confidence intervals and statistical hypotheses are then considered. The book concludes with a chapter on linear regression and correlation.

Since the book presupposes only college algebra, it is not written as a mathematical text, and proofs have not been presented. There are numerous exercises, and the author gives some evidence for the truth of certain theorems. However, some of the earlier chapters might have included more precise definitions. For example, there should be a more clear cut distinction between population parameters and sample characteristics. Perhaps the footnote on page 13 was considered adequate for this purpose. Again on page 72, the definition of mathematical expectation might be interpreted by the student as a theorem. Further there is no distinction between a cumulative distribution function and its sample image.

In dealing with tests of simple hypotheses, pages 130 to 135, the author properly states that the hypothesis and its test should be determined before the sample is observed. However, he is not as careful in his discussion of composite hypotheses. He even suggests on pages 142 and 143 that an hypothesis can be formulated after the sample values have been observed, and that the hypothesis can then be tested with that same sample. This practice is used all too frequently now, and the elementary books should stress the fact that if ob-

served data are to be used to suggest an hypothesis, the test should be made, not with those data, but with new data independently obtained.

Finally there are a few minor points. The subscript on s on page 149 should be $\bar{x}$, not $\tilde{x}$. On page 158, Problem 6.21, "five random samples" should read "a sample of five random values." An equality sign on page 207 is actually a minus sign. Tables IX and X are probability distributions of $|G|$, not G as defined on page 182. The left braces of the signs of aggregation on page 125 should follow the plus signs and not precede them. This reviewer believes the problems following sections 4.3 and 4.4 were inadvertently interchanged.

R. V. HOGG
State University of Iowa

Universal Mathematics, Part I. By the 1954 Summer Writing Group. Lawrence, Kansas, Student Union Book Store, 1954. x+310 pages. $2.75.

This preliminary edition is a planographed work which includes an experimental approach to a universal Freshman course. The motivation for writing this material has come from the Committee on the Undergraduate Program of the Mathematical Association of America, but the book is not an official publication of this committee. This volume is subtitled "Functions and Limits" and has the following chapter headings: I Coordinate Systems, II Scientific Measurement, III Functions, IV Limits, V Derivatives, VI Integrals, VII The Logarithm and Exponential Functions, VIII Summary. In each chapter the material is presented twice, first from an intuitive approach and then in a formal fashion.

Although intended for experimental teaching in a few institutions, this is not a textbook in the usual sense. It is best described in the words of the authors as "A book of experimental text materials." In the interest of speedy notice concerning this book, this brief review is being published in this issue; a more complete review will appear later.

C. B. ALLENDOERFER
University of Washington

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

THE TECHNICAL PUBLISHING SOCIETY

The Technical Publishing Society, a professional group to encourage the interchange of information among individuals engaged in preparing, editing, or

publishing technical and scientific documents, has been organized in Los Angeles. Membership includes editors, writers, illustrators, management and production personnel from industrial and governmental research and development laboratories. A Board of Directors is studying standards which will be used to accredit individual and chapter memberships.

For information write to Technical Publishing Society, P. O. Box 607, Tarzana, California.

PERSONAL ITEMS

The American Mathematical Society announces the appointment of Dr. J. H. Curtiss as Executive Director. He was formerly Adjunct Professor of Mathematics at New York University and Senior Scientist at the Institute of Mathematical Sciences of New York University.

Boston University announces: Dr. R. H. Brown of Columbia University has been appointed to an assistant professorship; Dr. S. L. Ross of Northeastern University has accepted an instructorship.

Central Michigan College of Education reports the following: Associate Professor L. G. Woodby of Mankato Teachers College, Minnesota, has been appointed to an associate professorship; Mr. D. R. Sudborough has been appointed Business Manager of the *Pentagon*; Miss Nikoline A. Bye has been elected President of the Michigan Council of Teachers of Mathematics.

Cornell University announces the following: Professor J. B. Rosser is Acting Chairman of the Department of Mathematics during 1954–55 while Professor R. J. Walker is on sabbatical leave and is at the University of California at Los Angeles; Dr. L. A. Rubel, previously a teaching assistant at the University of Wisconsin, has been appointed to an instructorship.

Massachusetts Institute of Technology reports: Dr. G. E. Baxter of the University of Minnesota, Mr. W. C. Fox, formerly an instructor at the University of Michigan, and Dr. Sigurdur Helgason of Princeton University have been appointed to C. L. E. Moore instructorships; Associate Professor R. J. Levit of the University of Georgia is on leave of absence from the University and has been appointed Visiting Assistant Professor; Mr. Bayard Rankin of the University of California has accepted a position as a research associate; Professor I. S. Cohen is on leave for the year 1954–55 and is a visiting member of the faculty of Columbia University; Professor C. C. Lin is a Guggenheim fellow at Cornell University and the California Institute of Technology.

At Syracuse University: Dr. Cyrus Derman, formerly of Columbia University, has been appointed to an instructorship; Assistant Professors W. R. Baum, P. W. Gilbert, and Erik Hemmingsen have been promoted to associate professorships.

United States Naval Academy announces: Assistant Professors R. W. Rector and H. K. Sokl have been promoted to associate professorships; Associate Professors N. H. Ball and A. E. Currier have been promoted to professorships;

Professor Ernest Hawkins has been promoted to the position of Senior Professor.

The University of Delaware announces that Mr. R. M. Lauer and Mr. W. G. Spohn have been appointed to instructorships.

University of Georgia reports the following: Dr. A. C. Cohen, Jr. was granted the $500 Michael Research Award for his work on truncated frequency distributions; Assistant Professor T. R. Brahana of Dartmouth College has been appointed to an assistant professorship; Miss Lola F. Kiser, previously a teaching assistant at the University, has been appointed to an instructorship.

University of Minnesota reports: Dr. Steven Orey of Cornell University and Dr. Irwin Fischer of Harvard University have been appointed to instructorships.

University of Mississippi announces the following: Dr. N. A. Childress and Mr. R. D. Sheffield are returning from graduate study to the University's Department of Mathematics as Assistant Professors; Assistant Professor L. L. Scott has been promoted to an associate professorship; Professor W. H. Spragens, Jr., is on leave from the University for an internship in General Education at the University of Chicago.

At the University of Pennsylvania: Dr. Wilhelm Stoll of the University of Tubingen, Germany, has been appointed Visiting Lecturer for one year beginning September 1954; Dr. Emil Grosswald, formerly an associate, has been promoted to an assistant professorship.

University of Western Ontario reports the following: Assistant Professor G. P. Henderson received a research award from the National Research Council of Canada; Mr. Arwel Evans, formerly a research student at Cambridge University, England, has been appointed to an instructorship.

Assistant Professor R. W. Ball of the University of Washington has accepted a position as Assistant Professor at Alabama Polytechnic Institute.

Assistant Professor Max Beberman of the University High School, University of Illinois, has been appointed to an associate professorship at Florida State University.

Professor David Blackwell of Howard University is on leave for the academic year of 1954–55 and is Visiting Professor at the Statistical Laboratory, University of California.

Mr. C. L. Bradshaw, formerly a mathematical analyst at Lockheed Aircraft, Marietta, Georgia, is now a member of the mathematics panel at Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Dr. C. C. Bramble of the Naval Proving Ground, Dahlgren, Virginia, has accepted a position on the technical staff of the Research and Development Division, Norden Laboratories, White Plains, New York.

Assistant Professor K. A. Bush of the University of Illinois has been appointed Head of the Mathematics Department of the University of Idaho.

Dr. Buchanan Cargal, formerly a senior aerophysics engineer at Consolidated Vultee Aircraft, Fort Worth, Texas, has accepted a position as a research engineer at Hughes Aircraft Company, Culver City, California.

Mr. R. E. Carroll, previously a dynamics engineer at Bell Aircraft Company, Niagara Falls, New York, is now a leader of the I.B.M. Computation Unit of the Company.

Dr. H. E. Chrestenson of Purdue University has been appointed to an assistant professorship at Whitman College.

Mr. R. C. Clelland, formerly an instructor at Hamilton College, has received an appointment as a research assistant at the University of Pennsylvania.

Mr. R. J. Cormier, recently in military service, is now a graduate student at University of Tennessee.

Mr. H. F. Davis, II, recently a part-time instructor at Massachusetts Institute of Technology, has been appointed Assistant Professor at Miami University.

Mr. W. R. Ferrante of Lafayette College has been promoted to an assistant professorship.

Dr. L. R. Ford, Jr., of Duke University has accepted a position with the Rand Corporation, Santa Monica, California.

Dr. G. E. Forsythe, formerly a mathematician with the National Bureau of Standards, Los Angeles, California, has a position as a research mathematician at the University of California at Los Angeles.

Mr. D. E. Freeland, recently a research assistant at the Statistical Laboratory of Purdue University, has accepted a position as a teaching assistant at Illinois Institute of Technology.

Dr. N. A. Goldsmith, previously head of the Mathematics Department of Henderson State Teachers College, Arkansas, has been appointed to a professorship at Chicago Teachers College.

Mr. Fred Gruenberger, formerly a computing service project supervisor at the University of Wisconsin, has a position as an operations analyst at the General Electric Company, Richland, Washington.

Mr. L. W. Gunter, previously a student at the University of Wisconsin, has been appointed to an instructorship at Western Michigan College.

Assistant Professor F. E. Hohn of the University of Illinois has been promoted to an associate professorship.

Dr. J. W. Hollingsworth, formerly an assistant in computing at the University of Wisconsin, has accepted a position as an engineer at the General Electric Company, Schenectady, New York.

Dr. C. W. Huff, teaching assistant at the University of Georgia, has accepted a position as Assistant Professor at Alabama Polytechnic Institute.

Mr. D. G. Johnson, recently a graduate assistant at Purdue University, is now a private in the U. S. Army.

Dr. Hyman Kamel of Cornell University has accepted an assistant professorship at Rensselaer Polytechnic Institute.

Mr. P. G. Kirmser of the University of Minnesota has been appointed to an associate professorship in the Department of Applied Mechanics, Kansas State College.

Assistant Professor S. R. Knox of Millsaps College has been promoted to an associate professorship.

Mr. E. L. Kretschmar, Jr., has a position as a mathematics teacher at Pasco High School, Dade City, Florida.

Mr. M. H. Lane, formerly a mathematical statistician for the Air Proving Ground, Eglin Air Force Base, Florida, is a graduate student at Alabama Polytechnic Institute.

Mr. H. S. Leonard, Jr., formerly a graduate student at Harvard University, has a position as a teaching fellow at the University.

Dr. Viktors Linis of the University of Saskatchewan has been appointed to an assistant professorship at the University of Ottawa.

Mr. J. P. Lipp, recently a student at the University of Oklahoma, is employed as an electrical engineer in the Advanced Electronic Center, General Electric Company, Ithaca, New York.

Mr. K. L. Loewen, previously an administrative assistant for Mennonite Aid Section, Mennonite Central Committee, Akron, Pennsylvania, has been appointed to an instructorship at Freeman Junior College.

Mr. Angelo Margaris, formerly a teaching assistant at Cornell University, has accepted an instructorship at Oberlin College.

Dr. D. C. B. Marsh, who has been a mathematics assistant at the University of Colorado, has been appointed to an assistant professorship at Texas Technological College.

Mr. H. H. Martens, previously a tester for Consolidated Edison Company of New York, New York City, is employed as a technical assistant with the Bell Telephone Laboratories, New York City.

Associate Professor W. S. Massey of Brown University is Visiting Associate Professor at Princeton University.

Associate Professor H. F. Mathis of the University of Oklahoma is now associated with Goodyear Aircraft Corporation, Akron, Ohio.

Mr. F. H. McGar, Jr., previously an instructor at Sweet Briar College, is now a demonstrator in the Department of Physics, Bryn Mawr College.

Professor W. G. McGavock is on leave of absence from Davidson College and is at the University of Wisconsin on a Ford Fellowship.

Mr. W. H. Mead, Jr., recently a student at the University of Southern California, is now a graduate student at the University of California.

Associate Professor E. J. Miles of Yale University has retired.

Dr. C. T. Molloy of the Vitro Corporation of America has accepted a position as staff engineer with the Lockheed Aircraft Corporation, Burbank, California.

Professor E. P. Northrop of the University of Chicago has resigned as Associate Dean of the College but retains his professorship. He will be on assignment from January 1, 1954 to July 1, 1955 by the University to the National

Science Foundation as consultant in the area of education in the sciences, and concurrently on assignment as consultant part-time to the Fund for the Advancement of Education.

Mr. F. P. Palermo, formerly a graduate assistant at Brown University, has been appointed to an instructorship at Princeton University.

Mr. D. S. Park, previously a mathematician at David Taylor Model Basin, Washington, D. C., is a mathematician with the Ordnance Corps, Letterkenny Ordnance Depot, Chambersburg, Pennsylvania.

Miss Mary Pettus of Sue Bennett College has accepted an assistant professorship at Union College, Barbourville, Kentucky.

Mr. R. S. Pieters, recently an instructor at Phillips Academy, has a position as Visiting Lecturer at Princeton University.

Mr. R. E. Priest, previously an assistant at the University of Illinois, is employed by the Bell Telephone Laboratories, New York City.

Mr. J. D. Reid, recently a mathematician for I.B.M. Corporation, Endicott, New York, is now at the U. S. Naval Training Center, Great Lakes, Illinois.

Mr. C. B. Rogers, formerly with the U. S. Navy, has a position as a hydrologic field assistant for the U. S. Geological Survey, University of New Mexico.

Mr. R. S. Roth, former graduate student at Carnegie Institute of Technology, is at Aberdeen Proving Ground, Maryland, as a mathematician.

Assistant Professor S. G. Roth, assistant to the dean of New York University's Division of General Education, has been named Assistant Director of the Office of the Budget of New York University.

Mr. Arthur Saastad, previously an industrial engineer for the U. S. Steel Company, Chicago, Illinois, has accepted a position as a research engineer at Convair Aircraft, San Diego, California.

Mr. C. T. Salkind, previously a teacher at S. J. Tilden High School, Brooklyn, New York, has been appointed to an assistant professorship at the Polytechnic Institute of Brooklyn.

Miss Ellen A. Sanders, formerly a teacher at Taylor County High School, Campbellsville, Kentucky, has been appointed Professor of Science at Montreat College.

Assistant Professor Clarence Schilling of the University of Tampa has accepted a professorship at Canisius College.

Mr. W. P. Shulevitz, recently a student at Wayne University, has joined the U. S. Army and is stationed at Fort Monmouth, New Jersey.

Rabbi Harold Shulman of Queens College has accepted a position as Mathematician with the Institute of Mathematical Sciences, New York University.

Mr. M. R. Simonson, formerly with the General Electric Company at Schenectady, New York, is now a technical engineer for General Electric Company, Evendale, Ohio.

Associate Professor M. B. Sledd of Georgia Institute of Technology has returned after spending a year's leave of absence at Massachusetts Institute of Technology.

Mr. E. C. Smith, Jr., of the University of Oregon has been appointed to a position as Assistant Professor at the University of Utah.

Professor Ernst Snapper of the University of Southern California is Visiting Professor at Princeton University.

Mr. A. D. Sollins, formerly a mathematician with the U. S. Coast and Geodetic Survey, Washington, D. C., has a position as Visiting Professor of Civil Engineering at the College of Agriculture and Mechanical Arts, University of Puerto Rico.

Dr. J. J. Sopka, previously a mathematician at Johns Hopkins University, is employed as Applied Science Representative, I.B.M. Corporation, New York City.

Assistant Professor O. E. Stanaitis of St. Olaf College has been promoted to an associate professorship.

Dr. C. J. Standish of Cornell University has been appointed Assistant Professor of Mathematics at Union College.

Mr. W. C. Styslinger, Jr., recently a student at Duquesne University, has accepted a position as a junior salesman with the I.B.M. Corporation, Pittsburgh, Pennsylvania.

Associate Professor S. L. Thorndike, head of the Mathematics Department of the College of Emporia, has been appointed Professor and Head of the Mathematics Department of Alma College.

Associate Professor W. J. Thron of Washington University has been appointed to an associate professorship at the University of Colorado.

Assistant Professor C. J. Tremblay of Bard College has been promoted to an associate professorship.

Mr. G. P. Weeg, a graduate assistant at Iowa State College, is employed as a mathematician by Engineering Research Associates, St. Paul, Minnesota.

Dr. C. G. Werner has been appointed to an instructorship at the University of Maine.

Assistant Professor R. H. Wilson, Jr., of the University of Louisville has accepted a position at the Naval Research Laboratory, Washington, D. C.

Mr. A. W. Yonda, formerly a graduate assistant at the University of Alabama, is at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland, as an assistant programmer.

Mr. B. K. Youse, recently an instructor at the University of Georgia, has been appointed to an assistant professorship at Emory University.

President Emeritus R. H. Reece of New Mexico Institute of Mining and Technology died on August 4, 1954. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ITINERARIES OF VISITING LECTURERS, 1954–55

The following itineraries have been arranged for the visiting lecturers under the program sponsored by the National Science Foundation and the Mathematical Association of America. The dates given are subject to slight adjustments. In case of regional arrangements, no attempt has been made to list all participating institutions or subdivide the time allocated.

R. H. Bing

Oberlin College	Oberlin, Ohio	Oct. 4–9
Univ. of Conn. and	Storrs, Conn.	Oct. 11–16
Trinity College	Hartford, Conn.	
Ohio Univ. and	Athens, Ohio	Nov. 1–3
Denison U.	Granville, Ohio	
So. Illinois U.	Carbondale, Ill.	Nov. 4–6
U. of Kansas and others	Lawrence, Kan.	Nov. 8–13
Vanderbilt U., Fisk U. & Geo. Peabody Tchs. Coll.	Nashville, Tenn.	Nov. 29–Dec. 3
The Florida State U.	Tallahassee, Fla.	Dec. 6–10
U. of Wisconsin (Milwaukee Ext.)	Milwaukee, Wis.	Jan. 3, 4
Carroll College	Waukesha, Wis.	Jan. 5, 6
Emmanuel Missionary College	Berrien Springs, Mich.	Jan. 10–12
State U. of Iowa	Iowa City, Ia.	Apr. 18, 19
U. of N. Dakota	Grand Forks, N. D.	Apr. 21–23

W. L. Duren, Jr.

Vassar College	Poughkeepsie, N. Y.	Oct. 19–21
U. of Buffalo	Buffalo, N. Y.	Oct. 25–30
U. of Alabama	University, Ala.	Nov. 1
Randolph-Macon, Sweet Briar and Lynchburg Coll.	Lynchburg, Va.	Nov. 9–12
Knox College	Galesburg, Ill.	Nov. 15–20
Southwestern at Memphis	Memphis, Tenn.	Jan. 11–15

T. Fort

Queens College	Charlotte, N. Car.	Feb. 1–2
North Carolina College	Durham, N. Car.	Feb. 4–5
Atlanta University and others	Atlanta, Ga.	Feb. 8, 9
College of William and Mary	Williamsburg, Va.	Feb. 10, 11
U. of Maryland	College Park, Md.	Feb. 14, 15, 16
State Teachers College	Shippensburg, Pa.	Feb. 18
Long Island U.	Brooklyn, N. Y.	Feb. 23, 24, 25
Mary Washington College	Fredericksburg, Va.	Feb. 28

G. Pólya

University of Redlands	Redlands, Calif.	Jan. 7–13
Los Angeles City College	Los Angeles, Calif.	Jan. 14
Arizona State College	Tempe, Ariz.	Jan. 27–Feb. 4
New Mexico A. and M.A.	State College, N. M.	Feb. 7–9
Utah State Ag. Coll.	Logan, Utah	Feb. 28–Mar. 2
U. of Colorado and environs	Boulder, Col.	Mar. 3–11
U. of Nebraska	Lincoln, Neb.	Mar. 14–18
St. Louis U.	St. Louis, Mo.	Mar. 21–23
U. of Oklahoma	Norman, Okla.	Mar. 24–30
Central State College	Edmond, Okla.	
Montana State College	Bozeman, Mont.	Apr. 10–17
Montana State U.	Missoula, Mont.	Apr. 17–20
U. of Idaho and	Moscow, Idaho	Apr. 21–30
Washington State College	Pullman, Wash.	
U. of Washington and environs	Seattle, Wash.	May 1–7
Williamette U.	Salem, Ore.	May 8–11
Oregon State College	Corvallis, Ore.	May 12–17
U. of Oregon	Eugene, Ore.	May 18–20

D. V. Widder

Syracuse U.	Syracuse, N. Y.	Mar. 28–31
U. of Kentucky	Lexington, Ky.	Apr. 1–7
U. of Georgia	Athens, Ga.	
Agnes Scott Coll. and	Decatur, Ga.	Apr. 8–14
Georgia Inst. of Tech.	Atlanta, Ga.	
Bowling Green U.	Bowling Green, Ohio	Apr. 21–23
Dartmouth Coll.	Hanover, N. H.	Apr. 25–30

THE FIFTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The fifteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 5, 1955. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not received a college degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, Box 30, Kent State University, Kent, Ohio, by a postcard request. Application blanks will be mailed out early in January. All applications must be filed with the Director not later than February 10, 1955. If three candidates are presented from a college or university, they are to constitute a team. If more than

three candidates are presented from any one college or university, the team of three must be named on the application. Fewer than three candidates from one college or university may compete as individual contestants.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to the contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200 and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $2000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on completion of the student's undergraduate work.

Reports of the fourteen previous competitions and examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942, October 1946, August–September 1947, December 1948, August–September 1949, 1950, 1951, October 1952, 1953 and 1954.

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: R. P. Agnew, J. P. Ballantine, Lipman Bers, Z. W. Birnbaum, David Blackwell, R. P. Boas, Jr., L. Carlitz, S. S. Chern, N. A. Court, Paul Cross, G. B. Dantzig, D. B. Dekker, W. Feller, Harley Flanders, Lester Ford, G. E. Forsythe, J. S. Frame, Michael Goldberg, Marshall Hall, S. T. Hu, Burrowes Hunt, A. S. Householder, P. S. Jones, J. L. Kelley, V. L. Klee, N. H. Kuiper, R. B. Leipnik, A. E. Livingston, G. W. Mackey, N. H. McCoy, L. H. McFarlan, W. E. Milne, L. J. Mordell, C. V. Newsom, S. Perlis, G. B. Price, W. V. Quine, G. de B. Robinson, Hans Samelson, M. R. Spiegel, R. D. Stalley, B. M. Stewart, R. R. Stoll, R. F. Tate, T. Y. Thomas, C. A. Truesdell, R. M. Winger, H. S. Zuckerman.

Mathematical Notes: G. E. Albert, R. Bellman, L. Bers, R. P. Boas, A. Brauer, L. E. Bush, L. Carlitz, S. Chowla, C. J. Coe, N. A. Court, R. R. Coveyou, J. M. Dobbie, F. G. Dressel, W. Feller, N. J. Fine, L. R. Ford, J. S.

Frame, B. Friedman, I. E. Garrick, H. Geiringer, W. Givens, A. W. Goodman, Harold Grad, H. C. Griffith, Marshall Hall, O. G. Harrold, Fritz Herzog, A. S. Householder, J. R. Isbell, S. L. Jamison, Mark Kac, L. M. Kelly, P. W. Ketchum, H. L. Lee, J. H. Lehner, Lee Lorch, C. C. MacDuffee, M. H. Martin, C. Masaitis, D. D. Miller, R. Moller, F. D. Murnaghan, L. Nirenberg, O. Ore, W. V. Parker, Harry Pollard, R. F. Rinehart, H. Robbins, H. N. Shapiro, V. L. Shapiro, S. Sherman, M. F. Smiley, W. S. Snyder, C. E. Springer, R. R. Stoll, G. Szegö, H. P. Thielman, A. W. Tucker, H. S. Vandiver, T. L. Wade, R. J. Walker, Morgan Ward, D. V. Widder, A. Wilansky.

Classroom Notes: H. W. Brinkmann, I. W. Burr, P. A. Clement, S. H. Crandall, R. B. Davis, Philip Franklin, H. K. Fulmer, Harriet Griffin, E. A. Hedberg, F. B. Hildebrand, R. C. James, R. E. Langer, Warren Loud, C. I. Lubin, F. H. Miller, R. K. Morley, J. C. Polley, Robert Solow, O. E. Stanaitis, F. M. Stewart, D. J. Struik, R. M. Thrall, J. R. Vatnsdahl, R. W. Walker, R. C. Yates.

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pennsylvania, April 30, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS, Fort Hays Kansas State College, Hays, March 26, 1955.

KENTUCKY

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 18–19, 1955.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, March 26, 1955.

MINNESOTA, College of St. Teresa, Winona, Minnesota, May, 1955.

MISSOURI, University of Kansas City, Spring, 1955.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1955.

NORTHERN CALIFORNIA, University of California, Berkeley, January 15, 1955.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA

ROCKY MOUNTAIN, University of Wyoming, Laramie, Spring, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May, 1955.

EMPLOYMENT OPPORTUNITIES

Southern Illinois University, Carbondale. Rank and salary depend upon qualifications. Candidates with Ph.D. or with M.A. and some additional graduate work will be considered.

University of Houston, Houston, Texas. Rank and salary depend on qualifications. Ph.D., Applied Math. or Geometry preferred. Math. Dept.

The MONTHLY is devoting this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specifix box number in care of the Mathematical Association of America, Buffalo 14, N. Y. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

Herbert Ellsworth Slaught Memorial Paper Number 3

Proceedings of the Symposium
on Special Topics in Applied Mathematics

Articles by: F. J. Weyl, Jean Leray, P. R. Garabedian, Edward McLeod, Jr., Martin Vitousek, A. H. Taub, M. Kac, D. E. Muller, S. S. Cairns, S. Chandrasekhar, E. W. Montroll. 73 pages. Paper.

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INDEX TO VOLUME 61, 1954

THE AMERICAN MATHEMATICAL MONTHLY

GENERAL MATHEMATICAL PAPERS

ALGEBRA

ALEXANDER, MARY K. See Thurston, H. S.

BRENNER, J. L. Linear Recurrence Relations, 171–173.

FRINK, ORRIN. Ideals in Partially Ordered Sets, 223–234.

GARCIA, MARIANO. On Numbers with Integral Harmonic Mean, 89–96.

HERSTEIN, I. N. A Note on Primitive Matrices, 18–20.

LAMBEK, J., and MOSER, L. Inverse and Complementary Sequences of Natural Numbers, 454–458.

MITCHELL, B. E. Unitary Multiples of a Matrix, 610–613.

MOSER, L. See Lambek, J.

SINGH, DALJIT. Concerning the Reciprocal of a Prime, 32–34.

THURSTON, H. S., and ALEXANDER, MARY K. The Equation $X^3+PX+Q=0$ in Binary Matrices, 8–17.

ZASSENHAUS, HANS. What is an Angle?, 369–378.

ANALYSIS

AISSEN, M. I. Some Remarks on Stirling's Formula, 687–691.

COWLING, V. F., and THRON, W. J. Zero-free Regions of Polynomials, 682–687.

EBERLEIN, W. F. The Elementary Transcendental Functions, 386–392.

GREEN, J. W. Recent Applications of Convex Functions, 449–454.

KENNEDY, E. C. Approximation Formulas for Elliptic Integrals, 613–619.

LORCH, LEE. The Principal Term in the Asymptotic Expansion of the Lebesgue Constants, 245–249.

OPPENHEIM, A. Criteria for Irrationality of Certain Classes of Numbers, 235–241.

RADEMACHER, HANS. On the Condition of Riemann Integrability, 1–8.

SCHWARTZ, J. The Formula for Change in Variables in a Multiple Integral, 81–85.

THRON, W. J. See Cowling, V. F.

APPLIED MATHEMATICS

CAIRNS, S. S. Computational Attacks on Discrete Problems, Sup. 29–31.

CHANDRASEKHAR, S. On Characteristic Value Problems in High Order Differential Equations Which Arise in Studies on Hydrodynamic and Hydromagnetic Stability, Sup. 32–45.

GARABEDIAN, P. R., MCLEOD, EDWARD, JR., and VITOUSEK, MARTIN. Recent Advances at Stanford in the Application of Conformal Mapping to Hydrodynamics, Sup. 8–10.

GOLDBERG, MICHAEL. Rotors within Rotors, 166–171.

HOHN, F. E. The Conference on Training in Applied Mathematics, 242–245.

KAC, M. Signal and Noise Problems, Sup. 23–26.

LERAY, JEAN. The Physical Facts and the Differential Equations, Sup. 5–7.

MACCOLL, L. A. Geometrical Properties of Two-dimensional Wave Motion, 96–103.

MCLEOD, EDWARD, JR. See Garabedian, P. R.

MONTROLL, ELLIOTT W. Frequency Spectrum of Vibrations of a Crystal Lattice, Sup. 46–73.

MULLER, D. E. Boolean Algebras in Electric Circuit Design, Sup. 27–28.

TAUB, A. H. Singularities on Shocks, Sup. 11–22.

VITOUSEK, MARTIN. See Garabedian, P. R.

WEYL, F. J. Introduction and Critique, Sup. 1–4.

EDUCATION

BRINKMANN, H. W. Mathematics in the Secondary Schools for the Exceptional Student, 319–323.
BUSH, L. E. The William Lowell Putnam Mathematical Competition, 542–549.
BUSH, R. R., MADOW, W. G., RAIFFA, HOWARD, and THRALL, R. M. Mathematics for Social Scientists, 550–561.
FULMER, H. K. Motives and Trends in Mathematics, 157–160.
JONES, B. W. Summer Conference in College Mathematics, 75–76.
KLINE, MORRIS. Freshman Mathematics as an Integral Part of Western Culture, 295–306.
LASLEY, J. W., JR. Comments on Mathematical Education, 378–380.
MACLANE, SAUNDERS. Of Course and Courses, 151–157.
MADOW, W. G. See Bush, R. R.
RAIFFA, HOWARD. See Bush, R. R.
THRALL, R. M. See Bush, R. R.
WEIL, ANDRÉ. Mathematical Teaching in Universities, 34–36.
WILLIAMS, W. L. What the Colleges are Doing about the Poorly Prepared Student, 86–88.

GEOMETRY, TOPOLOGY

BAGCHI, HARI DAS, and SARKAR, SHIB SANKAR. The Relationship between Harmonic and Anharmonic Collineations in a Plane, 397–401.
CLAWSON, J. W. A Chain of Circles Associated with the 5-line, 161–166.
DEKKER, DAVID. Twisted Curves and the Mean-Value Proposition, 607–610.
GANS, DAVID. Bounded Models of the Euclidean Plane—A Circular Model of the Euclidean Plane, 26–30.
GOLOMB, S. W. Checker Boards and Polyominoes, 675–682.
KELLY, P. J. Barbilian Geometry and the Poincare Model, 311–319.
MAY, K. O. Bounded Models of the Euclidean Plane—The Use of Condensed Graphs in Analytic Geometry, 31–32.
PÓLYA, G. An Elementary Analogue to the Gauss-Bonnet Theorem, 601–603.
ROBISON, G. B. Doodles, 381–386.
SARKAR, SHIB SANKAR. See Bagchi, Hari Das.
SWAIN, R. L. Bounded Models of the Euclidean Plane—Condensed Graphs, 21–26.
THÉBAULT, VICTOR. Concerning the Complete Quadrilateral, 604–606.
UNDERWOOD, R. S. Extended Analytic Geometry as Applied to Simultaneous Equations, 525–542.
VAUGHAN, H. E. Remark on My Paper "On Two Theorems of Plane Topology," 234.

HISTORY

BOYER, C. B. Carnot and the Concept of Deviation, 459–463.
GILLINGS, R. J. The So-Called Euler-Diderot Incident, 77–80.

PROBABILITY

HESS, F. G. Alternative Solution to the Ehrenfest Problem, 323–328.
KRUSKAL, MARTIN D. The Expected Number of Components under a Random Mapping Function, 392–397.
VON SCHELLING, HERMANN. Coupon Collecting for Unequal Probabilities, 306–311.

MATHEMATICAL NOTES

Edited by F. A. FICKEN, University of Tennessee

ABHYANKAR, S. S. Note on Positive Polynomials, 184–187.
ALDER, H. L. Note Concerning a Method for Finding Primes, 698.
BAKER, G. A., JR. Einstein Numbers, 39–41.
BEATTY, S. On Ruled Surfaces, 176–179.
BOTTEMA, O. Notes on the Skew Quadrilateral, 692–693.
BREUSCH, ROBERT. A Proof of the Irrationality of π, 631–632.
CAMPBELL, J. G., and GOLOMB, MICHAEL. On the Polynomial Solutions of a Riccati Equation, 402–404.
CARLITZ, L. A Note on Wolstenholme's Theorem, 174–176.
———. Congruences for the Number of n-gons formed by n lines, 407–411.
———. Note on the Cyclotomic Polynomial, 106–108.
COLES, W. J., and OLSON, F. R. A Note on Complete Residue Systems, 622.
ELLIS, DAVID. A Set-theoretic Description of Normal Topologies, 405–407.
ERDÖS, PAUL, and NIVEN, IVAN. The Number of Multinomial Coefficients, 37–39.

FAN, KY, and LORENTZ, G. G. An Integral Inequality, 626–631.
FRAME, J. S. Some Trigonometric Hyperbolic and Elliptic Approximations, 623–626.
GOLOMB, MICHAEL. See Campbell, J. G.
GOULD, H. W. A Note on a Paper of Grosswald, 251–253.
HEWITT, EDWIN. Remark on Orthonormal Sets in $\mathfrak{L}_2(a, b)$, 249–250.
JAMISON, FREE. Trisection Approximation, 334–336.
LIVINGSTON, A. E. A Necessary Condition for the Convergence of $\int_0^\infty f(x)dx$, 250–251.
LORENTZ, G. G. See Fan, Ky.
MCLEAN, DAVID. Idempotent Semigroups, 110–113.
MILES, E. P. Three Dimensional Harmonic Functions Generated by Analytic Functions of a Hypervariable, 694–697.
NANJUNDIAH, T. S. On a Formula of Grosswald, 700–702.
NEWTON, T. A. A Note on a Generalization of the Cauchy-Maclaurin Integral Test, 331–334.
NICOL, C. A. A Note Concerning the Quotient $(r^{p-1}-1)/p$, 562–563.
NIVEN, IVAN. See Erdös, Paul.
OLSON, F. R. The Non-existence of Rational Solutions for Certain Difference Equations, 179–181.
———. See Coles, W. J.
OPPENHEIM, A. Inequalities Connected with Definite Hermitian Forms, II, 463–466.
PARKER, W. V. Matrices and Polynomials, 182–183.
———. A Note on Normal Matrices, 330–331.
PROTTER, M. H. The Characteristic Initial Value Problem for the Wave Equation and Riemann's Method, 702–705.
RAO, S. K. LAKSHMANA. On the Evaluation of Dirichlet's Integral, 411–413.
SANDHAM, H. F. Two Series of Partitions, 104–106.
SCHOLOMITI, N. C. An Expression for the Euler ϕ-Function, 36–37.
SCOTT, W. R. Divisors of Zero in Polynomial Rings, 336
STEIN, S. The Fundamental Theorem of Algebra, 109.
THÉBAULT, VICTOR. Geometry of the Tetrahedron, 699–700.
———. Pascal Hexagons Associated with a Triangle, 328–330.
WARD, MORGAN. Cyclotomy and the Converse of Fermat's Theorem, 564.
WRIGHT, E. M. An Inequality for Convex Functions, 620–622.
YOUSE, B. K. Abridged Series for Numerical Evaluation, 187.

CLASSROOM NOTES

Edited by G. B. THOMAS, Massachusetts Institute of Technology

ANDREE, R. V., and PETERSON, G. M. Two Theorems on Groups, 641.
BALLANTINE, J. P. Note on Hero's Formula, 337.
BOAS, R. P., JR. Explaining the Line of Position, 417–418.
COLEMAN, A. J. The Probability Integral, 710–711.
DEKKER, DAVID B. A Paper Model of the Projective Plane, 113–115.
DUNCAN, D. G. Some Standard Problems in Integration Simplified, 421–422.
FULTON, C. M. Product of Determinants by Induction, 344–345.
GILLMAN, LEONARD. Remark on the General Polynomial of the Second Degree, 639–640.
GOHEEN, HARRY. On the Remainder in Formulas of Numerical Interpolation, Differentiation and Quadrature, 44–46.
GREENSTEIN, D. S. On the Iteration of Continuous Non-Increasing Functions, 637–639.
GUY, W. T., JR. Remark on a Graphical Procedure, 466–467.
HALFAR, EDWIN. Two Elementary Fallacies, 565.
HALPERIN, ISRAEL. A Fundamental Theorem of the Calculus, 122–123.
HENDLER, A. S. Relative Maxima and Minima of Functions of Two or More Variables, 418–420.
HOYT, J. P. A Natural Approach to the Fundamental Theorem of the Integral Calculus, 413–415.
KLAMKIN, M. S. A Geometric Determination of the Nature of the Roots of the Cubic, Biquadratic, and Quintic Equations, 340–342.
———. On a Graphical Solution of the First Order Linear Differential Equation, 565–567.
———. On the Vector Triple Product, 705–707.
KLAMKIN, M. S., and RUSSELL, J. P. On a Graphical Solution of a Class of Differential Equations, 188–189.
LEIMANIS, EUGENE. On the Motion of a Particle through a Resisting Medium of Variable Density, 117–119.
LOWELL, L. I. Comments on the Polar Planimeter, 467–469.
LUBIN, C. I. Vector Triple Product, 42–43.
MENGER, KARL. Tossing a Coin, 634–636.
MOISE, E. E. The Use of Induction in Existence Proofs, 192–193.
NAGLE, T. D. A Method for the Solution of the General Quartic, 343.
OSBORN, ROGER. A Consideration of Annuity Formulas, 708–709.
PAIGE, L. J. A Note on Indeterminate Forms, 189–190.

PARAMESWARAN, S. Skew-symmetric Determinants, 116.
PARKER, W. V. On an Application of the Vandermonde Determinant, 639.
PASCUAL, M. J. A Derivation of sin $(A-B)$ and cos $(A-B)$, 416.
PENNISI, L. L. A Rule for Solving a Certain Type of Exact Differential Equations, 633.
PERKINS, F. W. Invariants and the General Equation of the Second Degree, 191–192.
PETERSON, G. M. See Andree, R. V.
REID, W. P. On Numerical Solutions to the One-dimensional Wave Equation, 115.
RICHMOND, D. E. Trigonometry from Differential Equations, 337–340.
RUSSELL, J. P. See Klamkin, M. S.
SPIEGEL, M. R. Derivatives of Implicit Functions, 120–121.
STOLL, R. R. Determinants and Plane Equations, 255–256.
TRIFAN, D. Complex Roots of an Integral Rational Equation, 640–641.
WEINER, L. M. On the Computation of Definite Integrals, 254–255.
WILANSKY, ALBERT. Remarks on Yates' Note "Differentiating the Logarithm," 634.
YATES, R. C. Developments in Power Series, 256–258.
———. Differentiating the Logarithm, 120.

PROBLEMS AND SOLUTIONS

Edited by HOWARD EVES, University of Maine
and E. P. STARKE, Rutgers University

AUTHORS

Numbers refer to pages, boldface type indicating a problem solved and solution published; italics, a problem solved but the solution not published; ordinary type, a problem proposed.

Ablow, C. M., *50*, 477.
Agnew, R. P., **266**, **481**.
Aheart, A. N., *125*, *261*, *715*.
Aissen, M. I., *268*.
Aljančić, S., *267*.
Allan, D. W., 571.
Anderson, A. D., *649*.
Anderson, Jack, *49*.
Andress, W. R., *269*.
Anning, Norman, *125*, *131*, *653*.
Anselone, P. M., *49*, *124*.
Apostol, T. M., *574*, *650*.
Arnold, P. H., *348*.
Assmus, Edward, *55*.
Bagley, Robert, *651*.
Bailey, R. P., *203*, *425*.
Baker, Bonnie, *196*.
Baker, G. A., Jr., *203*.
Baker, I. N., **355**.
Baldwin, J. W., *196*, *197*, *260*, *261*, *348*, *424*, *425*.
Ballantine, J. P., 642.
Bankoff, Leon, 46, *49*, *125*, *131*, *195*, *196*, **198**, *260*, *261*, *346*, *426*, *474*, 476, 568, *653*, 711.
Barker, Trevor, *426*.
Barlaz, Joshua, *481*.
Bateman, Felice, *55*.
Bateman, P. T., *267*, *270*, **477**.
Baum, J. D., *55*.
Beil, R., **569**.
Bellman, Richard, *267*, 571.
Bennett, A. A., *354*.
Berberian, S. K., *53*.
Berndtson, Charles, *125*.
Berry, P. M., *260*, *643*, *645*, *712*.
Bickerstaff, T. A., *346*.
Bissinger, B. H., *714*, *715*.
Black, Tom, *475*.
Blatter, Christian, *715*
Bloch, René, 194, **644**.
Blundon, W. J., **646**, *648*, *650*, *653*, *721*, **722**.
Boas, R. P., Jr., *267*.
Boblétt, A. P., *346*.
Bojanić, Ranko, *267*.
Boron, L. F., *196*, *196*, *260*, *260*, *261*, *262*, *356*, *425*, *721*.
Bosgang, A. J., *49*.
Botsford, J. L., *424*, *425*, *426*.
Bowers, W. A., 642.
Boyer, R. H., **129**, *203*.
Brauer, G. U., *651*.
Brauer, Richard, *265*.
Braun, Julian, **124**, **195**, *196*, *260*, *261*, *346*, *347*, *348*, *426*, **473**.
Brennan, D. G., *712*.
Brenner, Joel, 345, *570*.
Breusch, Robert, *54*, **200**, **264**, *356*, *430*.
Briggs, W. E., *125*, *260*, *261*, *262*, **270**, *347*, *354*, *424*, *474*, *475*, *475*, *572*, *722*.
Brown, Bernice, *425*, *426*.
Brown, T. A., *53*, *55*, *354*, *433*.
Browne, D. H., *722*.
Buker, W. E., *125*, *195*.
Burke, J. F., **125**.
Campbell, M. O'N., *55*.
Campopiano, C. N., *128*, *715*.
Carlitz, Leonard, *53*, *197*, *202*, *204*, *205*, *265*, *266*, *270*, *347*, *356*, **431**, 476, **573**, *648*.
Carver, W. B., **47**, *49*, **49**, *198*, 345, *346*, *653*, **712**, *714*, *715*, **715**, **716**.
Caskey, R. L., *645*, *712*.
Charlesworth, G. B., *424*, *425*, *473*.
Cheney, W. F., Jr., **263**, *354*.
Cherry, W. J., **473**.
Chessin, Paul L., *125*, *197*, 258, *260*, *261*, *347*, *426*, *653*, *712*.

Chihara, T. S., **197.**
Churchill, E. P., *201.*
Clutterham, D. R., *196.*
Cohen, A. C., Jr., **718.**
Cohn, P. M., *354.*
Collins, G. E., *55.*
Conkling, R. M., *53, 55.*
Cothran, F. E., *55.*
Court, N. A., **424,** 642.
Courter, Richard, *124.*
Cowling, V. F., *482.*
Cox, J. R., *197.*
Coxeter, H. S. M., 572, **723.**
Crain, K. W., *261.*
Cullen, Helen F., *651.*
Currier, A. E., *203, 354.*
Cutler, E. H., **647.**
Danese, A. E., *475.*
Darraugh, J. E., *195, 645, 715.*
D'Atri, J. E., *125.*
Davis, M. V., *260.*
Demir, Hüseyin, *474,* 568, *643,* **643,** *645,* **713,** *717.*
Dempster, A. P., **269.**
DePicciotto, Elizabeth, *645.*
Dernham, Monte, *125, 261, 346, 712.*
Descartes, Blanche, **352.**
Deskins, W. E., *433.*
DeWan, Edmund, 470.
Dirac, G. A., 52.
Discepoli, Fred, *49, 125, 196, 196, 261.*
Dite, William, *715.*
Dodes, I. A., *196, 261, 715.*
Dougherty, Edgar, *197, 261.*
Douglas, William, *645, 712.*
Driscoll, R. J., *433.*
Duarte, F. J., *53, 261, 347, 425.*
Duncan, D. G., *650.*
Eisman, S. H., *260, 346, 347, 348, 475, 712.*
Ellis, D. O., **55,** *260, 651.*
Ellis, Kathryn, *55.*
Ellis, Robert, *651.*
Enterline, J. R., *196.*
Epstein, A. L., *49, 196, 196, 261, 475.*
Epstein, M. P., *425.*
Erdös, Paul, 52, **56,** 263, *265,* 350, 428.
Esser, Martinus, **570.**
Fan, Ky, 199.
Feldman, H. M., *53, 260, 260, 261, 425, 475.*
Fettis, Henry, *721.*
Ficken, F. A., *643.*
Fife, P. C., *712, 715.*
Fine, N. J., *53, 55, 56, 124, 260, 260, 261, 262, 347, 348, 648, 722.*
Finkbeiner, D. T., *55.*
Finkel, Daniel, *131.*
Fishback, W. T., *55.*
Flanders, Harley, *55,* 123, *128, 129,* **570,** *648,* **722.**
Fleishman, B. A., *649.*
Fleshler, A. D., *260, 260, 261.*
Ford, L. R., Jr., 46, *473.*
Ford, L. R., *346.*
Foreman, Calvin, *56.*
Forsythe, G. E., *470.*
Freitag, A. H., *346.*
Freitag, Herta, *346.*
Freund, J. E., *433.*
Frucht, Robert, **650.**
Fulk, L. A., *261, 348.*
Fulks, W. B., *650.*
Furstenberg, Harry, *48, 49, 49, 53, 55, 56, 129, 129, 267, 433, 650.*
Gaddum, J. W., *55.*
Gale, E. I., *474.*
Gamzon, Jean, *645, 712.*
Gamzon, W. V., *644, 712.*
Garreau, G. A., 126.
Gehman, H. M., **49.**
Gleason, A. M., *53,* **53.**
Godard, Russell, *49.*
Goering, Orville, *49.*
Goffman, Casper, 350.
Goheen, Harry, *55, 425,* 477, *650.*
Going, Mildred, *125.*
Goldberg, Karl, *54, 204.*
Goldberg, Michael, *196, 570, 714.*
Goldman, Oscar, *265.*
Golomb, S. W., 427.
Goormaghtigh, R., **652.**
Gordon, R. M., 259.
Gould, H. W., *49, 53, 125, 266, 654, 712.*
Gould, Katherine, *433.*
Gould, S. H., *270,* **355, 569.**
Grace, A. G., *125.*
Grace, E. E., *55.*
Grant, A. S. G., *425, 645.*
Grant, W. T., *645, 712.*
Graue, L. C., 259, 345, *714, 719.*
Greatrix, E. K., *715.*
Green, Simon, 642.
Greenspan, Donald, *124.*
Greenspan, Norman, **353,** *475, 643.*
Greenstein, D. S., *197, 260, 261,* **347,** *354, 570, 648, 650.*
Greenwood, R. E., *260, 348, 712.*
Gregg, C. V., *129,* **130.**
Gregory, Arthur, *50, 131, 197, 260, 260, 261.*
Greville, T. N. E., *712.*
Grinstein, Louisa S., *643, 645, 715.*
Gross, Oliver, *354.*
Grosswald, Emil, *129, 260, 261, 649, 721.*
Gruenberger, Fred, *712.*
Guderley, Karl, *721.*
Gugenheim, C., *53.*
Gunderson, N. G., *53.*
Gutzman, R. R., *196.*
Guy, R. K., *712, 716.*
Haber, Seymour, *354.*
Haggard, J. D., *196, 645, 712.*
Hamblen, J. W., *195, 196, 260, 261, 262, 712.*
Harary, Frank, *260.*
Hardy, J. W., Jr., *650.*
Harris, V. C., **719.**
Hatcher, J. R., 124, *196, 197, 260, 261,* **262,** *571,* **643.**
Hauer, H. J., *712.*
Hausman, B. A., *196,* 712.
Haynes, T. H., Jr., *53.*
Heath, R. W., *196.*

Heller, Joy, *712*.
Hendler, A. S., 711.
Hennessy, W. P., *645*.
Henry, Boyd, *712*.
Herbst, R. T., *721*.
Herschbach, Dudley, *261*.
Herwitz, P. S., *55*.
Herzog, Fritz, *54*, *127*.
Hilton, H. K., *712*.
Hodge, P. G., Jr., *715*.
Hoehn, M. H., *261*.
Hoffman, Walter, *196*.
Hoggatt, Vern, *49*, *53*, *125*, *125*, *196*, *196*, *197*, *261*, *346*, *426*, 568, *644*, *645*, 646, *650*.
Holdridge, Douglas, *49*, *125*, *125*, *196*, *197*.
Hood, R. T., *346*, *712*.
Horton, R. E., *712*.
Horváth, J., *55*.
Howell, J. M., *348*, *475*.
Huff, R. W., *197*, *346*.
Hurt, J. M., *55*.
Hyde, A. R., *49*, *49*, *50*, *125*, *125*, *195*, *196*, *196*, *260*, *261*, *346*, *347*, *348*, *424*, *425*, *426*, *474*, *475*, **569**, *644*, *645*, *650*, *712*, *715*, *716*.
Indritz, Jack, *55*.
Ivkovic, Zoran, *260*.
Iyer, R. Venkatachalam, *131*, *647*, 652.
Jackson, J. R., *643*, *643*.
Jacobson, Bernard, *347*.
James, Walter, *423*.
Jamison, Robert, *49*.
Jamison, S. L., *267*, *433*.
Jasper, S. J., *125*.
John, P. W. M., *55*.
Johnson, D. L., 477, *648*.
Johnson, E. S., *712*.
Johnson, P. B., 423, **424**.
Johnson, Philip, *55*.
Johnson, Virginia, *645*, *712*.
Jones, John, Jr., *50*, *125*, *196*, *196*, *260*.
Jurgensen, Ray, *196*, *196*, *261*.
Kac, Mark, *265*.
Karns, C. W., *347*.
Karst, Edgar, 194, *644*, *645*, *712*.
Kelly, J. B., *270*, *353*, *433*, *650*.
Kenner, M. R., *131*.
Kenny, B. C., *53*, *55*.
Kerr, C. E., *197*, *260*, *261*.
Kingston, J. M., *712*.
Kirchberg, Allen, *125*.
Kirchberg, M. A., **645**, *653*.
Kirk, Sybil, *196*.
Kirmser, P. G., *129*, *267*, *268*, *354*, *475*, *650*, *715*.
Kissling, Robert, 52, *432*.
Kitamura, M., *125*.
Klamkin, M. S., *49*, *49*, *50*, *53*, *124*, *125*, *125*, *128*, *129*, *196*, *196*, *197*, *199*, *203*, *260*, *260*, *261*, 263, *267*, *347*, *348*, 350, 423, *425*, *426*, 427, 470, *475*, *475*, *569*, *571*, *649*, 711, *714*, *715*, *719*, *721*.
Klee, V. L., Jr., *197*.
Klimczak, W. J., *125*.
Klopfenstein, R., *346*
Kneale, S. G., *354*.
Knödel, Walter, *354*.
Kocher, Diane, *645*.
Kravitz, Sam, *195*, *346*.
Kravitz, Sidney, *125*, *197*, *645*.
Laden, H. N., *424*.
Lander, L. J., 423.
Langr, Josef, *424*, *643*.
Lariviere, Rose, *261*.
Leghorn, J. G., *196*.
Lehner, Joseph, *197*, 646, *650*, **653**, 718.
Leifer, H. R., *712*.
Leja, S., *49*, *124*, **354**, *650*.
Lemke, C. E., *53*, *55*.
Leum, Mark, *261*.
Levenspiel, Octave, *473*, *712*.
Lewis, D. C., *650*.
Linis, Viktors, 259, *425*, *475*, *650*, *716*.
Lipsich, H. D., *426*.
Littlejohn, T. C., *55*, *433*.
Livingston, A. E., **48**, *49*, *55*, *124*, *125*, 128, *196*, *197*, *197*, *260*, *260*, *261*, 264, *267*, *348*, *475*, *482*, *650*.
Lorch, Lee, *475*.
Lorentz, G. G., **205**, 428, *649*.
Lubin, C. I., 194, *643*.
Lundell, A. T., *55*.
Lyness, R. C., *131*.
MacPhail, M. C., 428.
Maggs, P. B., *473*.
Maier, E. A., *267*.
Mandelbaum, David, *49*, *260*.
Manheim, Jerome, *260*.
Manheim, Sylvia, *125*.
Marcus, Marvin, *570*.
Marer, Fred, *125*.
Marquis, R. H., **48**, *473*.
Marsh, D. C. B., *49*, *50*, *54*, *55*, *125*, *195*, *196*, *196*, *197*, *260*, *261*, *354*, *424*, **425**, *426*, 429, *433*, *473*, *474*, *570*, *643*, *644*, *645*, *650*, *653*, *712*, *715*.
Martin, Beckham, *48*, *125*, *196*, *474*, *714*.
Martins, H. H., *261*.
Mason, R. M., 124.
Massera, José L., 645.
Mattis, Don, *645*.
McConnon, Myles, *356*, *474*.
McCracken, D. D., *125*.
McKay, J. H., *433*.
McShane, E. J., *197*.
Mead, L. V., *196*, *198*, *346*, *473*.
Means, J. H., *645*, *712*.
Meisters, G. H., *474*.
Mendelsohn, N. S., 198, *267*.
Menkes, H. R., *260*.
Mercer, R. J., *196*, *196*.
Merriman, G. M., *426*.
Mesner, D. N., *648*.
Mewborn, Ancel, *196*.
Michalup, Erich, 194, *260*, *653*, *719*.
Miller, C. E., **196**.
Miller, Donald, *196*.
Miller, G. T., *261*.
Miller, H. C., Jr., *55*.
Miller, J. D., *260*, *261*, *426*, *474*.
Miller, Norman, *128*, *129*, *426*, *426*.

Miller, Raphael, *54*, 55.
Millman, George, *49*, *125*.
Milner, E. C., *270*.
Milosevich, Kovina, *55*, **261**, *347*.
Mitchell, B. E., *712*.
Monsky, Paul, 123, *570*.
Moore, Marian, *196*.
Morduchow, Morris, *196*, *196*, *260*, *261*, *721*.
Morris, S. L., *196*.
Morrow, T. M., *196*.
Moser, Leo, *127*, *260*, **262**, *270*, *354*.
Moser, William, *49*.
Mott, George, *712*.
Motzkin, T. S., **433**.
Mulcrone, T. F., *712*, *715*.
Mumford, D. B., *473*, *474*, *643*, *645*.
Murnick, Sharon, *125*.
Murtha, John, *712*.
Musch, E. J., *125*, *196*, *712*.
Muskat, David, *645*, *712*.
Muskat, J. B., *125*, *261*, *424*, *645*, *712*.
Myers, E. F., *346*.
Nakamura, Yoshiv, *54*.
Newman, D. J., 51, *129*, 199, 350, **351**, *354*, 572, 646, *648*, *650*, 718, *720*.
Nickel, J. A., *260*.
Nordhaus, E. A., *49*, *50*, *125*.
Norton, D. A., *433*.
Novikoff, Albert, *268*.
O'Callaghan, Celestine, *721*.
Ogilvy, C. S., *50*, *124*, *125*, *196*, *261*, *261*, *346*, 470, *473*, *570*, *643*, *712*, *714*, *715*, *716*, *717*.
Olds, C. D., *347*, 711, *719*.
Oliphant, M. W., *643*.
Oliver, H. W., *354*.
Olmsted, Margaret, *49*, *125*, *196*, *197*, *197*.
Olson, F. R., **347**.
Oppenheim, Alexander, **471**.
Oppenheim, J. H., *55*.
Osner, H. J., *712*.
Ostrowski, A. M., *260*.
Pai, M. V., *261*.
Pancoe, Arthur, *261*.
Parameswaran, S., *124*, *125*, *125*, *129*, *196*, *197*, *260*, *261*, *261*, *481*, *647*, *650*, *653*, *719*.
Park, Bart, *475*.
Parker, F. D., *54*, *55*, *196*, *197*, *475*, *715*.
Parker, W. V., *131*.
Parrish, G. B., *124*, *125*, *125*, *196*, *196*, *197*.
Pascual, M. J., *49*, *50*, *125*, *196*, *196*, *197*, *260*, *261*, *261*, *347*, *348*, *644*, *653*, *721*.
Pavley, R. F., *55*.
Pease, D. K., *55*.
Peery, B. F., **643**.
Pennell, W. O., *49*.
Penney, Walter, *426*, *644*, *645*, *712*.
Pennington, J. V., 470, *712*.
Pennisi, L. L., *49*, *53*, *124*, *196*, *197*, *203*, *261*, *426*, *475*, *570*.
Perel, W. M., *54*.
Periasastri, M., *124*, *125*.
Perisho, C. R., *473*.
Phelps, C. R., *354*.
Phelps, R. R., *432*.
Pinzka, C. F., **346**, *347*, *348*, *425*, **475**, **643**.
Piranian, George, **124**, *649*, *650*.
Pitcairn, Joel, *651*.
Piza, P. A., *347*, *712*.
Plait, Alan, *712*.
Pollard, Harry, 264.
Popiel, N., Jr., *712*.
Portman, W. O., *55*.
Pracher, K., *352*.
Preston, G. B., *55*, *351*.
Ramler, O. J., *714*.
Rao, K. Subba, *131*.
Raynor, G. E., *648*.
Read, R. C., *347*, *348*.
Redheffer, R. M., **430**.
Reeves, R. F., *129*, *354*.
Reich, Edgar, *203*, 427.
Reid, J. D., *197*, *261*, *261*.
Resnikoff, H. W., *473*.
Rhoades, B. E., *197*, *261*, *475*.
Rhodes, A. P., *644*.
Richardson, D. P., **261**.
Ringenberg, L. A., 47, *49*, *50*, *124*, *125*, *125*, *195*, *196*, *196*, *197*, *261*, *424*, *426*, *473*, *474*, **474**, *475*, *475*, 568, **648**, *649*, **649**, *653*, *712*, *715*, *719*.
Roberts, G. G., *645*
Robertson, M. S., *480*.
Robinson, D. W., *55*.
Robinson, H. A., *125*, **572**.
Robinson, L. B., *721*.
Rodriguez, A. J., *54*, *55*, *129*.
Rosenbaum, Joseph, 126, *131*.
Rosenberg, Max, *712*.
Rosenfeld, Azriel, *49*, *50*, *55*, *195*, *196*, *197*, *261*, *346*, *348*, *354*, **425**, *645*, *651*, *712*.
Rothman, David, *474*, *645*, *712*, 715.
Royster, W. C., **479**.
Rozycki, E. P., *712*.
Russell, D. C., *347*, *425*, *475*, *715*.
Rutledge, W. A., *54*, *55*.
Saaty, Thomas, *473*.
Sanders, W. M., *712*, *715*.
Sandham, H. F., 263, **265**, 718.
Sasakawa, Tatuya, *49*.
Sasser, D. W. *55*, **351**.
Saul, William, *712*.
Saworotnow, P. P., *570*.
Schäffer, J. J., *651*.
Schillo, Paul, *425*.
Schoenberg, I. J., 199, 476.
Scholomiti, N. C., *196*.
Schwid, Nathan, *196*, *196*, *197*.
Schwerdtfeger, H. W. E., *650*.
Scibor-Marchocki, R. I., *721*.
Scott, W. R., *651*.
Selfridge, J. L., 423.
Serbyn, W. D., *197*, *260*, *261*, *261*, *262*, 423.
Seward, D. M., *198*.
Shafer, R. E., *125*, *196*, *196*, *262*, *653*, *712*, *714*, **714**.
Shapiro, H. S., **54**, 126, *267*, 571, *650*.
Shelton, S. W., *712*.
Shepherd, W. L., *426*.
Shiflett, L. J., *125*.
Shklov, N., *197*, *475*, *712*, *714*.

Shugart, S. H., *196*.
Siddiqi, Omar Ali, **720**.
Simester, J. H., *643*.
Singer, James, *55*.
Skalskyj, Michael, *49*, *650*.
Small, William, *55*, *261*, *348*.
Smith, F. C., *261*.
Smith, G. C., *260*, *261*.
Smith, H. R., 572.
Smith, Malcolm, *712*, *715*.
Smith, R. P., *261*, *261*.
Spiegel, M. R., **52**, *56*, 126, 350, *425*, 568, *721*.
Spong, R. A., *712*.
Sprague, R. H., *55*.
Stallard, F. W., *55*.
Stamey, W. L., *55*.
Stanaitis, O. E., *49*, **50**, *53*, *56*, *124*, *125*, *129*, *196*, *197*, *202*, *205*, **260**, **260**, *261*, *266*, *268*, *356*, *425*, *426*, *430*, *481*, *572*, *575*, *649*, *650*, *654*, *715*, *720*, 721.
Stanović, B., *721*.
Starke, E. P., *347*.
Steger, A., *261*.
Stein, R. B., *712*.
Steinberg, E. J., *261*, *261*.
Steinberg, Robert, 264.
Stewart, B. M., *201*, **348**.
Sudborough, D. R., *55*, *196*, *261*, *474*.
Sumner, T. H., *196*.
Swanson, C. A., *125*, *346*.
Sylwester, A. V., *49*, *196*, *197*.
Tapscott, R. P., *712*.
Taussky, Olga, *55*, *570*.
Thébault, Victor, 194, *198*, 345, *424*, 470, *643*, 646, *647*.
Thielman, H. P., *129*.
Thomas, G. H. M., *55*, *426*, *651*.
Thomas, J. D., *125*.
Thompson, C. W., *645*.
Tierney, J. A., *196*, *196*, *261*, *262*, *643*.
Tingley, A. J., *473*, *712*.
Todd, John, *56*.
Treuenfels, Peter, *54*, *55*, 259, *715*.
Trigg, C. W., *125*, *195*, **195**, *261*, *261*, 345, *426*, *474*, *645*, *712*.
Trimble, G. R., Jr., *650*.
Trotter, H. F., *424*.
Trumpler, Donald, *196*, *197*.
Trumpler, T. A., *55*.
Tuckerman, Bryant, **127**.
Umberger, E. H., *648*.
Underwood, F., *643*, *643*, *721*.
Ungar, Peter, *56*, *353*.
Ury, Hans, *475*.
Utz, W. R., *348*.
Valentine, F. A., 264.
Van Voorhis, W. R., **47**, *125*.
Vaughan, H. E., **651**.
Vause, R. Z., *125*, *195*, *196*, *196*, *475*, *475*, *569*.
Von Schelling, Hermann, *572*, **574**, **650**.
Wahab, J. H., *261*.
Walker, A. W., *50*, **51**.
Walker, G. W., *197*, *653*, *712*.
Walker, Ineatha, **643**.
Wallach, Robert, *712*.
Walsh, E. G., *712*.
Walter, R. M., *197*.
Wang, Chih-yi, *49*, *49*, *53*, **55**, **127**, *129*, *196*, *261*, *266*, *347*, *425*, *475*, **571**, *643*, *649*, *650*, *653*, *654*, **713**, *715*, **720**, *721*.
Ward, J. A., *196*.
Ward, L. E., Jr., *197*, *261*, **348**, *425*, *426*, *475*.
Ward, L. E., Sr., **47**, *425*.
Ward, Morgan, *650*.
Ward, W. D., *645*, *712*.
Warga, Jack, *574*.
Webster, M. S., 123, *569*.
Weiner, Daniel, *346*.
Weiner, L. M., *54*, *55*.
Weissblum, W. E., 572.
Wendel, J. G., *267*.
Wesson, J. R., *54*.
Whittaker, J. V., *49*, *54*, *55*, *125*, *129*, *197*, **202**, **204**, *260*, *261*, *266*, **425**, **432**, **475**, **649**, *650*.
Wicht, M. C., *197*.
Wilansky, Albert, *124*, 126, *129*, *267*, *475*, *482*, 642, 719.
Wild, R. E., *475*.
Wilf, H. S., *55*, *425*.
Wilf, W. S., *125*.
Wilkins, J. E., Jr., *55*, *129*, **203**, *267*, *715*.
Willcox, A. B., *55*.
Willey, Maud, *196*.
Williams, R. R., Jr., *261*.
Wilson, R. H., Jr., *261*, *346*, *712*.
Wing, G. Milton, 718.
Winkelman, J. R., *475*.
Wiser, H. C., *55*.
Woods, Roscoe, *474*, *570*.
Wormser, Arthur, *197*.
Wright, E. M., *267*, **268**, *354*, *433*, 480.
Wyzalek, Monica, *645*, *712*.
Yos, Jerrold, *55*.
Young, J. W., *125*, *196*.
Zimmerberg, H. J., *203*.
Zirakzadeh, A., *48*.

SOLUTIONS

Numbers in boldface type refer to problems, those in lightface to pages.

E-804, 568. **E-1051**, 471. **E-1066**, 47. **E-1067**, 48. **E-1068**, 49. **E-1069**, 49. **E-1070**, 50, 471. **E-1071**, 124. **E-1072**, 124. **E-1074**, 125. **E-1076**, 195. **E-1077**, 195. **E-1078**, 196. **E-1079**, 197. **E-1080**, 198. **E-1081**, 260. **E-1082**, 260. **E-1083**, 261. **E-1084**, 261. **E-1086**, 346. **E-1087**, 346. **E-1088**, 347, 471. **E-1089**, 348. **E-1090**, 348. **E-1091**, 423. **E-1092**, 424. **E-1093**, 424. **E-1094**, 425. **E-1095**, 426. **E-1096**, 472. **E-1097**, 473. **E-1098**, 474. **E-1099**, 474, **E-1100**, 475. **E-1101**, 569. **E-1102**, 570. **E-1103**, 570. **E-1104**, 570. **E-1106**, 643. **E-1107**, 643. **E-1108**, 643. **E-1109**, 644. **E-1110**, 644. **E-1111**, 712. **E-1112**, 713. **E-1113**, 714. **E-1114**, 715. **E-1115**, 716.

3651, 428. **4459**, 477. **4501**, 52. **4502**, 53. **4503**, 54. **4504**, 54, 351. **4505**, 55. **4506**, 56. **4507**, 126. **4508**, 127, 264. **4509**, 128. **4510**, 129. **4511**, 130. **4512**, 200. **4513**, 202. **4514**, 203. **4515**, 203. **4517**, 205. **4518**, 264. **4519**, 265. **4520**, 266. **4522**, 268. **4523**, 269. **4524**, 270. **4525**, 351. **4526**, 352. **4527**, 353. **4528**, 354. **4529**, 355. **4531**, 430. **4532**, 431. **4533**, 432. **4534**, 433. **4535**, 479. **4536**, 480. **4537**, 481. **4539**, 572. **4541**, 573. **4542**, 574. **4543**, 646. **4544**, 647. **4545**, 648. **4546**, 649. **4547**, 649. **4548**, 651. **4549**, 652. **4550**, 652. **4551**, 653. **4552**, 719. **4553**, 720. **4554**, 720. **4556**, 721. **4557**, 722. **4558**, 722.

RECENT PUBLICATIONS

Edited by E. P. VANCE, Oberlin College

NEW BOOKS RECEIVED

135–136, 275–276, 493, 578–579, 658.

REVIEWS

Names of authors are in ordinary type, those of reviewers, in capitals.

Bardell, R. H., and Spitzbart, Abraham. *College Algebra*, M. A. OLIVER, 270–271.
Bellman, Richard. *Stability Theory of Differential Equations*, J. J. L. HINRICHSEN, 358.
Bieberbach, Ludwig. *Theorie der Geometrische Konstruktionen*, E. S. HAMMOND, 62–63.
Britton, J. R., and Snively, L. C. *College Algebra*, J. C. POLLEY, 271–272.
Brown, C. H. *The Teaching of Secondary Mathematics*, J. H. HLAVATY, 59–60.
Caris, V. B. *Elements of Algebra*, J. C. POLLEY, 272–273.
Degrazia, Joseph, *Math is Fun*, S. W. MCINNIS, 357.
Eves, Howard. *An Introduction to the History of Mathematics*, P. S. JONES, 433–435.
———. See Newsom, C. V.
Fischer, C. H. See Rider, P. R.
Fryer, H. C. *Elements of Statistics*, R. V. HOGG, 725–726.
Funkhouser, H. G. See Weeks, A. W.
Goedicke, Victor. *Introduction to the Theory of Statistics*, G. I. BUTTERBAUGH, 359–360.
Goffman, Casper. *Real Functions*, C. E. RICKART, 134–135.
Gruenberger, Fred. *Computing Manual, Third Edition*, GLENN LEWIS, 133–134.
Householder, A. S. *Principles of Numerical Analysis*, W. H. DURFEE, 655–656.
Johnson, R. E. *First Course in Abstract Algebra*, A. L. FOSTER, 654–655.
———, McCoy, N. H., and O'Neill, Anne F. *Fundamentals of College Mathematics*, V. H. WELLS, 58.
Langer, R. E. *Ordinary Differential Equations*, T. S. PETERSON, 437.
MacDuffee, C. C. *Theory of Equations*, L. CARLITZ, 723–724.
McCoy, N. H., See Johnson, R. E.
Menger, Karl. *Calculus, A Modern Approach*, H. E. BRAY, 483–492.
Middlemiss, R. R. *Algebra for College Students*, H. S. GRANT, 274.
———. *College Algebra*, H. S. GRANT, 273–274.
Newsom, C. V., and Eves, Howard. *Introduction to College Mathematics*, A. H. DIAMOND, 724–725.
1954 Summer Writing Group. *Universal Mathematics, Part I, Functions and Limits*, C. B. ALLENDOERFER, 726.
O'Neill, Anne F., See Johnson, R. E.
Rassweiler, I. See Rassweiler, M.
Rassweiler, M. and Rassweiler, I. *Fundamental Procedure of Financial Mathematics*, J. O. BLUMBERG, 60–61.
Rider, P. R., and Fischer, C. H. *Mathematics of Investment*, J. O. BLUMBERG, 60–61.
———. *Plane Trigonometry*, FRED ROBERTSON, 656–657.

Randolph, J. F. *Trigonometry*, FRED ROBERTSON, 656–657.
Rosenbach, J. B., and Whitman, E. A. *Intermediate Algebra for Colleges*, H. S. GRANT, 274–275.
Smail, L. L. *Analytic Geometry and Calculus*, J. H. BLAU, 435–437.
———. *Mathematics of Finance*, J. O. BLUMBERG, 60–61.
Snively, L. C. See Britton, J. R.
Spitzbart, Abraham. See Bardell, R. H.
Staff of the Bateman Manuscript Project: Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, Francisco G. Tricomi. *Higher Transcendental Functions*, C. TRUESDELL, 576–578.
Struik, D. J. *Elementary Analytic and Projective Geometry*, T. K. PAN, 131–133.
Thomas, G. B. *Calculus*, W. FRASER, 575.
———. *Calculus and Analytic Geometry*, J. H. BLAU, 435–437.
Thron, W. J. *Theory of Functions of a Complex Variable*, W. D. MUNRO, 206.
Tintner, Gerhard. *Mathematics and Statistics for Economists*, SAMUEL GOLDBERG, 360–362.
Weeks, A. W. and Funkhouser, H. G. *Plane Trigonometry*, FRED ROBERTSON, 656–657.
Whitman, E. A. See Rosenbach, J. B.
Wold, Herman. *Demand Analysis*, C. B. WINSTEN, 61–62.

NEWS AND NOTICES

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GENERAL INFORMATION

California Conference for Teachers of Mathematics, 277.
Conference on the Role of Women's Colleges in the Physical Sciences, 580–581.
Cooperative Graduate Summer Sessions in Statistics, 65.
International Scholars Forum, 362.
M. I. T. Special Summer Sessions, 438.
Mathematics in Public High Schools, 65.
Meeting of the Mathematics Division of A.S.E.E., 276–277, 579–580.
National Register of Scientific Personnel, 136–137.
New Magazine for High School Mathematics Students, 64.
Numerical Analysis Research U.C.L.A., 658–659.
Preliminary Actuarial Examinations Prize Awards, 659–660.
Recordings of Radio Programs of the University of Oklahoma, 278.
Report of Manpower Resources in Mathematics, 580.
Research Fellowships in Psychometrics, 580.
Sale of Mathematical Library, 494.
Sixth Annual Institute for Teachers and Professors of Mathematics, 207.
Society for Industrial and Applied Mathematics, 660.
Summer Conference for High School Teachers, 207.
Summer Conferences in 1954, 76.
Summer Conferences for Teachers, 438.
Summer Sessions, 278–281, 362–363, 438–439.
Symposium on Monte Carlo Methods, 137.
The Technical Publishing Society, 726–727.

NECROLOGY

Berry, E. M., 215.
Brandeberry, J. B., 72.
Briant, R. C., 497.
Brown, H. S., 497.
Coolidge, J. L., 365.
Corliss, J. J., 286.
Darling, F. W., 585.
Daum, J. A., 497.
Dickson, L. E., 365.
Dresden, Arnold, 497.
Eiten, Rev. R. B., 144.
Findlay, William, 498.
Hardy, J. G., 72.
Holl, D. L., 498.
Irwin, O. W., 144.
Johnson, R. A., 440.
Kalicki, Jan, 286.
Lamb, R. C., 72.
Livingston, G. R., 215.
MacNeish, H. F., 72.
Mason, R. H., 144.
McDonald, J. H., 144.
McKinsey, J. C. C., 72.
Pehrson, E. W., 663.
Post, E. L., 498.
Reece, R. H., 732.
Rosenbaum, Joseph, 144.
Sanders, J. E., 498.
Smith, C. W., 215.
Smith, R. F., 440.
Stephens, R. P., 663.
Stetson, J. M., 498.

REPORTS AND ANNOUNCEMENTS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS AND ANNOUNCEMENTS OF THE ASSOCIATION

Fifteenth Annual William Lowell Putnam Mathematical Competition, 734–735.
Itineraries of Visiting Lecturers, 1954–55, 733–734.
New Sectional Governors of the Association, 498.
Officers and Committees as of January 1, 1954, 221–222.
Report of the Treasurer for the Year 1953, 286–288.
The 1954 Combined Membership List, 499.
Thirty-fifth Summer Meeting of the Association, H. M. GEHMAN, 664–667.
Thirty-seventh Annual Meeting of the Association, H. M. GEHMAN, 216–220.
Visiting Lectureship Program, 440–441.

MEETINGS OF ITS SECTIONS

Allegheny Mountain Section, May 1954 Meeting, L. T. MOSTON, 587–588.
Illinois Section, May 1954 Meeting, A. WAYNE MCGAUGHEY, 588–591.
Indiana Section, May 1954 Meeting, J. C. POLLEY, 591–592.
Iowa Section, April 1954 Meeting, FRED ROBERTSON, 513–516.
Kansas Section, April 1953 Meeting, LAURA Z. GREENE, 290–291.
Kentucky Section, May 1954 Meeting, W. J. ROBINSON, 593–595.
Louisiana-Mississippi Section, February 1954 Meeting, Z. L. LOFLIN, 366–368.
Maryland-District of Columbia-Virginia Section, December 1953 Meeting, C. H. FRICK, 293–294.
Maryland-District of Columbia-Virginia Section, May 1954 Meeting, C. H. FRICK, 595–596.
Metropolitan New York Section, March 1954 Meeting, E. MARIE HOVE, 501–503.
Michigan Section, March 1954 Meeting, S. D. CONTE, 503–505.
Minnesota Section, October 1953 Meeting, F.C. SMITH, 146–148.
Minnesota Section, May 1954 Meeting, F. C. SMITH, 667–669.
Missouri Section, May 1954 Meeting, MARGARET F. WILLERDING, 597–599.
Northern California Section, January 1954 Meeting, C. D. OLDS, 443–444.
Ohio Section, April 1954 Meeting, FOSTER BROOKS, 516–519.
Oklahoma Section, October 1953 Meeting, R. V. ANDREE, 148–149.
Pacific Northwest Section, June 1954 Meeting, J. MAURICE KINGSTON, 671–673.
Philadelphia Section, November 1953 Meeting, G. C. WEBBER, 291–293.
Rocky Mountain Section, April 1954 Meeting, F. M. CARPENTER, 519–522.
Southeastern Section, March 1954 Meeting, H. A. ROBINSON, 505–513.
Southern California Section, March 1954 Meeting, P. H. DAUS, 445–447.
Southwestern Section, April 1954 Meeting, R. L. WESTHAFER, 522–524.
Upper New York State Section, May 1954 Meeting, N. G. GUNDERSON, 669–671.
Wisconsin Section, May 1954 Meeting, SISTER MARY FELICE, 599–600.

PERSONAL INFORMATION

Newly elected members of the Association, 72–74, 145–146, 288–290, 441–442, 499–501, 585–587.

The following persons presented papers at meetings of the Association and its Sections.

Ainsworth, Sidney, 600.
Aissen, M. I., 294.
Alexander, H. W., 592.
Alger, P. L., 671.
Anderson, A. G., 588.
Arena, F. J., 147.
Arnoff, E. Leonard, 517.
Arsove, M. G., 673.
Artin, Emil, 509, 511.
Atchison, W. F., 655.
Ayre, H. Glenn, 589.
Bagley, R. W., 513.
Bankoff, Leon, 446.
Barrow, D. F., 511.
Bartholomew, W. V., 514.
Beaver, R. A., 670.
Beberman, Max, 588.
Beenken, May M., 447.
Begle, E. G., 218.
Bell, Clifford, 445.
Bellman, R. E., 446.
Bernhart, Arthur, 148.
Bing, R. H., 600.
Blackwell, David, 596.
Blake, R. G., 511.
Block, H. D., 669.
Blumenthal, L. M., 598.
Bonnell, C. R., 668.
Bradley, Eugene, 593.
Brauer, Alfred, 510.
Briggs, W. E., 520, 521.
Brinkman, H. W., 218.
Bruck, R. H., 514.
Brumfield, Emalou, 517.
Bryant, B. F., 509.
Buck, C. C., 512.
Bush, R. R., 218.
Bussey, W. H., 668.
Butchart, J. H., 523.
Butz, R. K., 519.
Cain, Ethel, 515.
Callahan, Ethel B., 670.
Calloway, J. M., 148.
Cameron, E. A., 507, 594.
Cameron, J. M., 596.
Chellevold, J. O., 516.
Childress, N. A., 513.
Chowla, S., 520, 521.
Churchill, R. V., 665.
Cogan, E. J., 587.
Cohen, A. C., Jr., 511.
Coleman, A. J., 670.
Cook, Laurence, 600.
Cook, R. H., 520, 520.
Cornelison, J. B., 594.
Cothran, F. E., 508.
Cowan, R. W., 507.
Coyle, I. F., 599.
Crosby, D. R., 673.
Currie, J. C., 507.

Deal, R. B., 149.
Drucker, B. M., 510.
Duren, W. L., Jr., 218, 368, 600, 668.
Dwyer, P. S., 505.
Dye, L. A., 512.
Eatherton, L. J., 520.
Eaves, J. C., 594.
Ellis, D. O., 512.
Ellis, Wade, 517.
Eschrich, Mary A., 593.
Farrell, E. J., 444.
Fettis, H. E., 517.
Fort, M. K., Jr., 513.
Fort, Tomlinson, 507, 508, 510.
Fowler, K. A., 524.
Fox, A. H., 671.
Fox, R. H., 292.
Frank, D. H., 502.
Fuller, L. E., 291.
Fulmer, H. K., 507.
Fulton, C. M., 443.
Gaddum, J. W., 504.
Gentry, F. C., 524.
Georges, J. S., 590.
Goheen, H. E., 515.
Goldberg, Samuel, 518.
Gonzalez, M. O., 509.
Gormsen, S. T., 512.
Graham, Beulah, 594.
Griffin, F. L., 368, 368.
Gurland, John, 515.
Haas, Leonard, 600.
Hadnot, B. F., 509.
Hall, D. W., 294.
Harary, Frank, 504.
Hassler, J. O., 149.
Hebel, I. L., 522.
Hendrickson, M. S., 523.
Hoffman, A. J., 595.
Hoffman, J. E., 149.
Horton, T. R., 512.
Hoyt, J. P., 294.
Huff, G. B., 308.
Huffer, R. C., 600.
Huskey, H. D., 445.
Hutchison, C. A., 599.
Jackson, E. J., 598.
Jackson, S. B., 596.
Jaeger, C. G., 446.
James, R. D., 673.
Jamison, Free, 443.
Johnson, W. M., 590.
Jones, B. W., 218, 521.
Jones, John, Jr., 366.
Juberg, Richard, 668.
Kanter, L. H., 367.
Karp, Carol R., 523.
Kaufmanis, Karlis, 146.
Kelley, J. L., 290.
Kempner, A. J., 520, 521.
Kirkham, Don, 514.
Kuhn, H. W., 292.
Lange, L. H., 592.
Langman, Harry, 504.
LaSalle, Margaret M., 367.
Lasley, J. W., Jr., 506.
Latta, G. E., 443.
Leech, J. S., 532.
Lehmer, D. H., 665.
Leipnik, R. B., 673.
Lessard, Roger, 670.
Lewis, G. R., 588.
Little, E. P., 505
Lonseth, A. T., 665.
Loomis, L. H., 665, 665, 665.
Lorentz, G. G., 504.
Lotkin, M. M., 596.
MacLane, Saunders, 218.
Madow, W. G., 218.
Martin, Beckham, 518.
Mayor, J. R., 600.
McLaughlin, J. J., 600.
McMillan, Brockway, 502.
Meserve, B. E., 589.
Mesner, D. M., 504.
Mewborn, A. B., 444.
Michel, R. J., 599.
Mientka, W. E., 520, 521.
Miller, F. H., 502.
Mishoe, Luna, 596.
Mitchell, B. Ernest, 367.
Mitchell, B. Evans, 366.
Moore, J. T., 508.
Morrill, D. E., 367.
Morris, F. R., 444.
Mount, B. H., Jr., 587.
Murphy, C. H., Jr., 294.
Nering, E. D., 524.
Newman, Morris, 596.
Newsom, C. V., 218.
Nicholson, S. B., 523, 523.
Niven, Ivan, 673.
Nohel, J. A., 508.
Novak, J. D., 508.
Oakley, C. O., 293, 672.
Olive, Gloria, 592.
Ostrofsky, Morris, 505.
Otteson, Elli, 600.
Overholtzer, G. K., 508.
Pall, Gordon, 590.
Palmer, T. P., 592.
Parker, W. V., 507.
Patterson, P. B., 508.
Pennisi, L. L., 590.
Perry, N. C., 511.
Peterson, G. M., 148.
Phipps, C. G., 506.
Plunkett, R. L., 512.
Popken, Jan, 669.
Powell, J. H., 505.
Poritsky, Hillel, 670
Preston, C. C., 147.
Price, G. B., 218.
Pugsley, D. W., 594.
Punga, Valdemars, 671.
Purcell, E. J., 524.
Rado, Tibor, 592.
Raiffa, Howard, 218.
Rawlins, C. H., Jr., 444.
Razgunas, Leo, 504.
Recht, A. W., 522.
Reichelderfer, P. V., 665.
Rider, P. R., 517.
Robertson, Fred, 516.
Robinson, Gilbert deB., 518.
Robinson, L. V., 518.
Rogers, C. A., 520.
Rohde, F. Virginia, 510.
Rose, D. C., 594.
Ross, Alan, 515.
Royster, W. C., 512.
Rutland, Leon, 519.
Rutledge, W. A., 510.
Saltzer, Charles, 518.
Scherk, Peter, 665.
Seeney, J. A., 598.
Seybold, M. Anice, 589.
Shanks, E. B., 509.
Shanks, M. E., 592.
Sister Mary Nicholas, 291.
Smith, C. B., 511.
Smith, N. B., 514.
Smith, R. G., 291.
Smith, S. R., 521.
Sobczyk, Andrew, 524.
Spears, O. S., 149.
Spencer, G. L., II, 294.
Stabler, E. R., 503.
Stanaitis, O. E., 146.
Stevenson, Guy, 593.
Stewart, B. M., 665.
Stone, W. M., 672.
Story, Frances, 598.
Straus, E. G., 446.
Sward, Dorothy, 600.
Taam, Choy-tak, 294.
Temple, V. B., 367.
Thrall, R. M., 218.
Tillotson, D. B., 672.
Tinbergen, Jan, 292.
Tompkins, C. B., 445, 665.
Trifan, Deonisie, 524.
Trigg, C. W., 447.
Tucker, A. W., 218.
Varner, Walter W., 521.
Wagner, Laura, 600.
Wahab, J. H., 506.
Wallace, A. D., 368.
Ward, J. A., 593.
Webb, D. L., 524.
Weinstock, Robert, 444.
Weitkamp, Harvey, 517.
Wend, D. V. V., 514.
Wendt, Arnold, 590.
Wilk, M. B., 515.
Wilson, R. H., Jr., 594.
Winkelman, J. R., 515.
Wolf, F. L., 668.
Woodley, W. L., 147.
Wyler, Oswald, 523.
Wylie, C. R., Jr., 521.
Wyman, Max, 665.
Yeardley, Nelson, 587.
Young, J. W., 507.
Youse, B. K., 511.

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